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Applications of Integration: Arc Length

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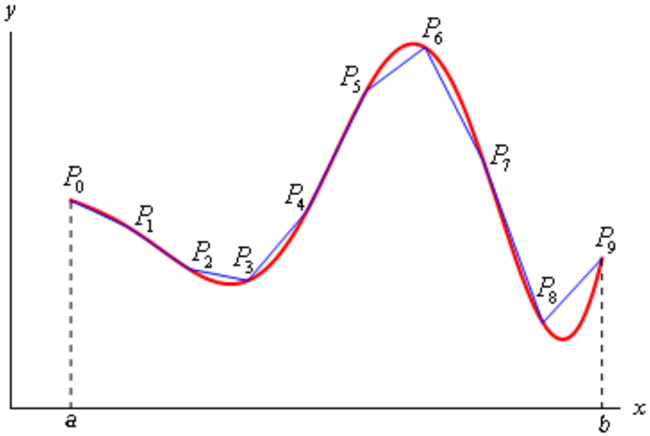
Arc Length of the Curve $y = f(x)$

- In this section, we are going to look at computing the arc length of a function.
- We want to determine the length of the continuous function $y = f(x)$ on the interval $[a, b]$. We'll also need to assume that the derivative is continous on $[a, b]$.
- Initially we'll need to estimate the length of the curve by dividing the interval up into n **equal subintervals** each of width Δx and we'll denote the point on the curve at each point by P_i.
- We can then approximate the curve by a series of straight lines connecting the points.

Arc Length of the Curve $y = f(x)$

Here is the sketch of this situation for $n = 9$.



(source: *https://tutorial.math.lamar.edu*)

Denote the length of each of these line segments by $|P_{i-1}\ \ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ \ P_i|$$

Arc Length of the Curve $y = f(x)$

We can get the exact length by taking n larger and larger. In other words, the exact length wil be,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} \quad P_i|$$

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Arc Length of the Curve $y = f(x)$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define

$$\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$$

and

$$\Delta x = x_i - x_{i-1}$$

Then, we can compute directly the length of the line segments as follows.

$$|P_{i-1}\ \ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the **Mean Value Theorem** we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

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Arc Length of the Curve $y = f(x)$

Therefore, the length can now be written as,

$$|P_{i-1} \quad P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$
$$= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2}$$
$$= \Delta x \sqrt{1 + \left[f'(x_i^*)\right]^2}$$

The exact length of the curve is then,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1} \quad P_i|$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \sqrt{1 + \left[f'(x_i^*)\right]^2}$$

Arc Length of the Curve $y = f(x)$

However, using the **definition of the definite integral**, this is nothing more than,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

A slightly more covenient notation is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Arc Length of the Curve $x = h(y)$

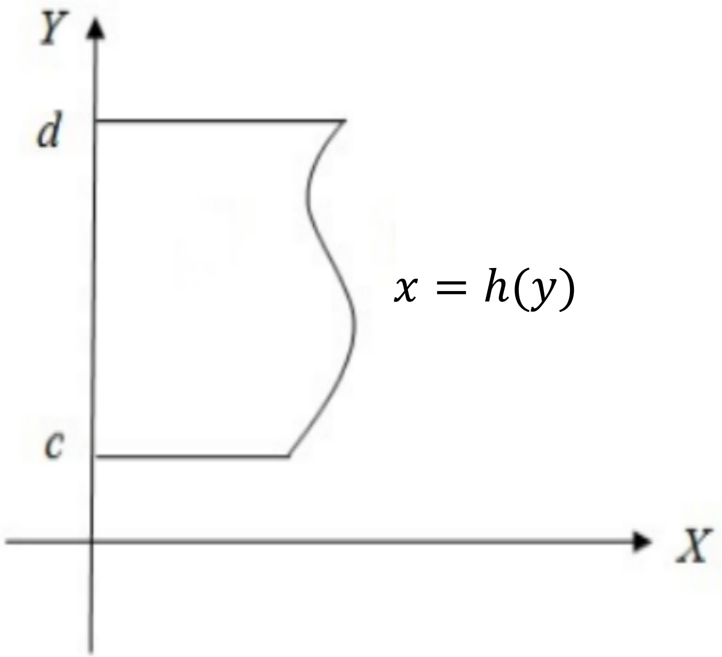
In a similar fashion we can also derive a formula for $x = h(y)$ on $[c, d]$. This formula is,

$$L = \int_c^d \sqrt{1 + [h'(y)]^2}\ dy$$

Again, a slightly more covenient notation is the following.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Arc Length of the Curve $x = h(y)$



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Summary

Arch length can be computed by,

$$L = \int ds$$

Where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx, \qquad \text{if } y = f(x)\text{, and } a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy, \qquad \text{if } x = h(y)\text{, and } c \le y \le d$$

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Example 1

Determine the length of $y = \ln(\sec x)$ between $0 \leq x \leq \frac{\pi}{4}$.

Solution:

In this case we'll need to use the first ds (in the summary) since the function is in the form $y = f(x)$. So, let's get the derivative out of the way

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x$$

so

$$\left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

Let's also get the root out the way since there is often simplification that can be done and there's no reason to do that inside the integral

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

Example 1

Solution (continuation):

Note that we could drop the absolute value bars here since secant is positive in the range given.

The arc length is then,

$$L = \int_0^{\frac{\pi}{4}} \sec x \; dx$$

$$= [\ln|\sec x + \tan x|]_0^{\frac{\pi}{4}}$$

$$= \ln\left(\sqrt{2} + 1\right)$$

Example 2

Determine the length of $x = \frac{1}{2}y^2$ for $0 \leq x \leq \frac{1}{2}$. Assume that y is positive.

Solution:

We'll use the second ds for this one as the function is already in the correct form for that one. Also, the other ds would again lead to a particularly difficult integral. The derivative and root will then be,

$$\frac{dx}{dy} = y \qquad \Longrightarrow \qquad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$$

Before writing down the length, notice that we were given x limits and we will need y limits for this ds. With the assumption that y is positive these are easy enough to get. All we need to do is plug x into our equation and solve for y. Doing this give us

$$0 \leq y \leq 1$$

Example 2

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\ dy$$

This integral will require the following trigonometry substitution.

$$y = \tan\theta \quad \Rightarrow \quad dy = \sec^2\theta\ d\theta$$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$

$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1 + y^2} = \sqrt{1 + \tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

The length is then,

$$L = \int_0^{\frac{\pi}{4}} \sec^3\theta\ d\theta$$

$$= \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right)$$

Arc Length in Parametric Equations

In this section we will look at the arc length of the parameteric curve given by,

$$x = f(t), \qquad y = g(t), \qquad \alpha \leq t \leq \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from α to β. We will also need to assume that the curve is traced out frem left to right as t increases. This is equivalent to saying

$$\frac{dx}{dt} \geq 0 \qquad \text{for} \qquad \alpha \leq t \leq \beta$$

So, let's start out the derivation by recalling the arc length formula as we first derived previously (summary in page 10)

Arc Length in Parametric Equations

We will use the first ds (summary in page 10) because we have a nice formula for the derivative in terms of the parametric equations. To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}\,dt$$

The arc length formula then becomes,

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\ \frac{dx}{dt}\,dt = \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\ \frac{dx}{dt}\,dt$$

Factor out the denominator from the square root, we have

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ \frac{dx}{dt}\,dt$$

Arc Length in Parametric Equations

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will alow us to cancel the two derivatives that are outside the square root and this gives

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Notice that we could have used the second formula for ds (summary in page 10) above if we had assumed instead that

$$\frac{dy}{dt} \geq 0 \qquad \text{for} \qquad \alpha \leq t \leq \beta$$

If we has gone this route in the derivation we would have gotten the same formula.

Summary (Arc Length for Parametric Equations)

Given the parametric equations

$$x = f(t), \qquad y = g(t), \qquad \alpha \leq t \leq \beta$$

Arc length for parametric equations can be computed by,

$$L = \int_{\alpha}^{\beta} ds$$

where

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example 3

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t),$$

$$y = 3\cos(t),$$

where $0 \leq t \leq 2\pi$.

Solution:

We know that this is a circle of radius 3 centered at the origin. It will be traced out exactly once in this range. So, we can use the formula for parametric equations. We'll first need the following,

$$\frac{dx}{dt} = 3\cos(t)\,, \qquad \frac{dy}{dt} = -3\sin(t)$$

Example 3

Solution (continuation):

The length is then,

$$L = \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\ dt$$

$$= \int_0^{2\pi} 3\sqrt{\sin^2(t) + \cos^2(t)}\ dt$$

$$= 3\int_0^{2\pi} dt$$

$$= 6\pi$$

Example 4

Use the arc length formula for the following parametric equations.

$$x = 3\sin(3t),$$

$$y = 3\cos(3t),$$

where $0 \leq t \leq 2\pi$.

Solution:

Notice that this is the identical circle that we had in the Example 3 where the arc length is 6π. However, for the range given we know it will trace out the curve three times instead once as required for the formula.

In this case the derivatives are,

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

Example 4

Solution (continuation):

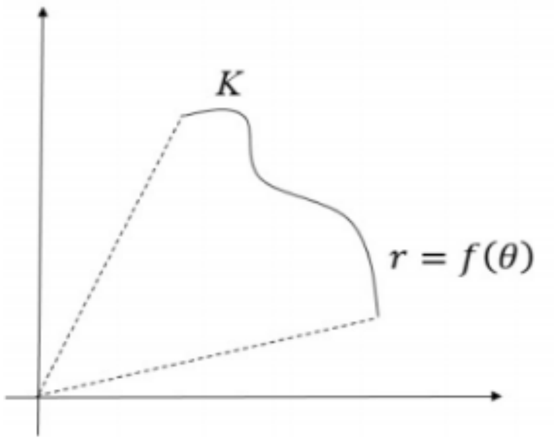
so, the arc length formula gives,

$$L = \int_0^{2\pi} \sqrt{81\sin^2(3t) + 81\cos^2(3t)}\ dt$$
$$= \int_0^{2\pi} 9\ dt$$
$$= 18\pi$$

The answer we got from the arc length formula in this example is 3 times the actual length in Example 3.

Arc Length in Polar Coordinate System

In this section, we want to compute the arc length L for polar curve $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$ and f is differentiable on $[\alpha, \beta]$.



The arc length L can be computed by assuming the parametric equations as follow

$$x = r\cos\theta, \qquad y = r\sin\theta$$

where $r = f(\theta)$, and $\alpha \leq \theta \leq \beta$.

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Arc Length in Polar Coordinate System

Next, we have

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta,$$
$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta.$$

Then,

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\sin\theta\cos\theta\frac{dr}{d\theta} + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\sin\theta\cos\theta\frac{dr}{d\theta} + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

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Arc Length in Polar Coordinate System

Based on the previous formula (summary in page 18) we have

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \; d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \; d\theta$$

Summary (Arc Length in Polar Coordinate System)

Given a polar curve $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$ and f is differentiable on $[\alpha, \beta]$.

The arc length in polar coordinate system can be computed by,

$$L = \int_{\alpha}^{\beta} ds$$

where

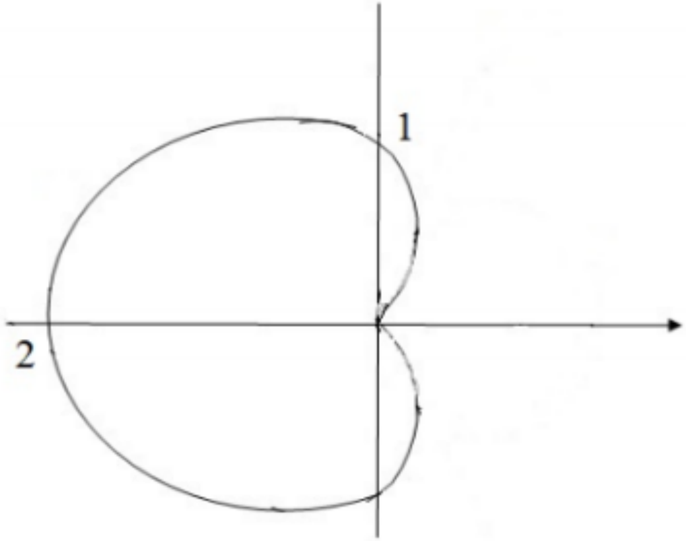
$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\ d\theta$$

Example 5

Determine the arc length of the cardioid $r = 1 - \cos\theta$,

where $0 \leq \theta \leq 2\pi$.

Solution:



Example 5

Solution (continuation):

$$L = \int_0^{2\pi} ds$$

$$= 2\int_0^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\ d\theta$$

$$= 2\int_0^{\pi} \sqrt{(\sin\theta)^2 + (1-\cos\theta)^2}\ d\theta$$

$$= 2\int_0^{\pi} \sqrt{2\cdot 2\sin^2\frac{1}{2}\theta}\ d\theta$$

$$= 4\int_0^{\pi} \sin\frac{1}{2}\theta\, d\theta = -4\left[2\cos\frac{1}{2}\theta\right]_0^{\pi}$$

$$= -4(0-2) = 8$$

Derivative of the Arc Length

Derivative of the arc length is the ds for each cases.

- Case 1: (summary in page 10)
- Case 2 (parametric equations): (summary in page 18)
- Case 3 (polar coordinate system): (summary in page 26)

Quiz 1

Will be announced in eLOK after class (check this out).

The deadline of submission: today, April 26th at 17.30 WIB

Thank You