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Applications of Integration: Area of a Surface of Revolution

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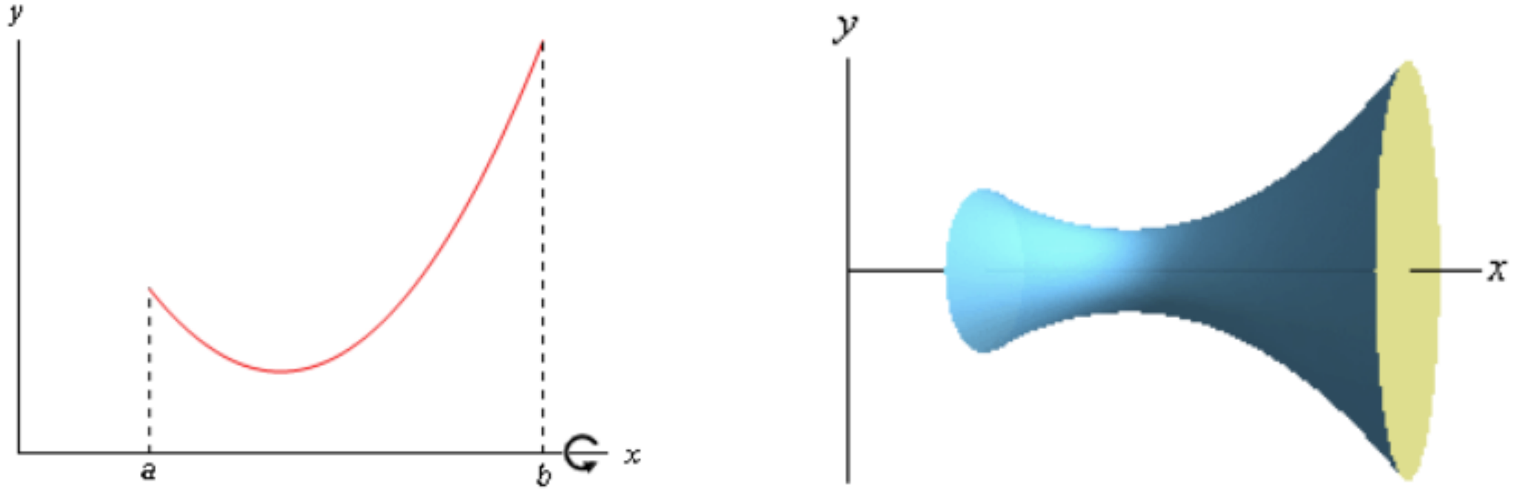
Area of Surface of Revolution

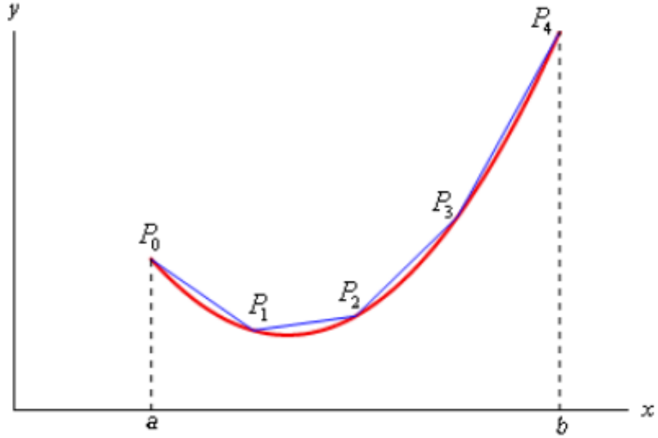
- In this discussion, we are going to look once again at **solids of revolution**. We want to find the surface area of the region.
- Let's look at rotating the continuous function $y = f(x)$ in the interval $[a, b]$ around the x-axis. We'll also need to assume that the derivative is continuous on $[a, b]$. The following is a sketch of a function and the solid of revolution we get by rotating the function around the x-axis.



(source: *https://tutorial.math.lamar.edu*)

Area of Surface of Revolution

- We can derive a formula for the surface area much as we derived the formula for **arc length**.
- We'll start by dividing the interval into n equal subintervals of width Δx.
- On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of each subinterval.
- Here is the sketch of that for our representative function using $n = 4$.



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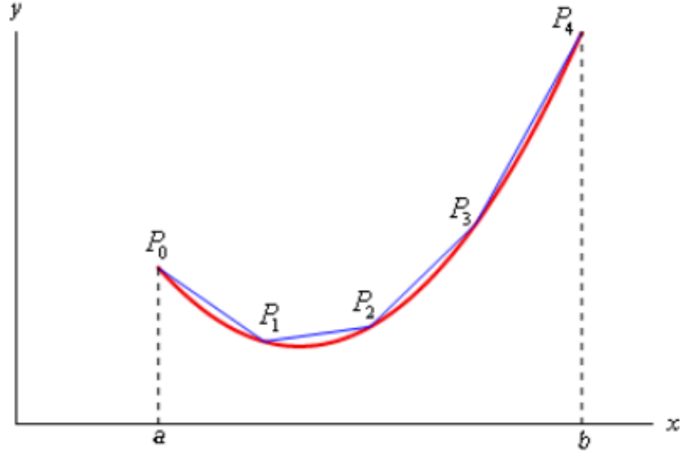
Area of Surface of Revolution



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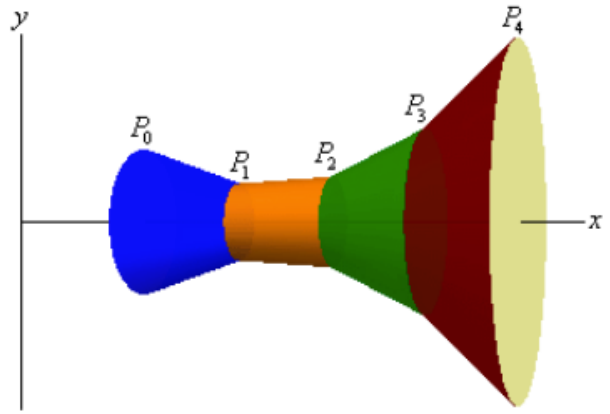
Now, rotate the approximatons around the x-axis and we get the following solid.



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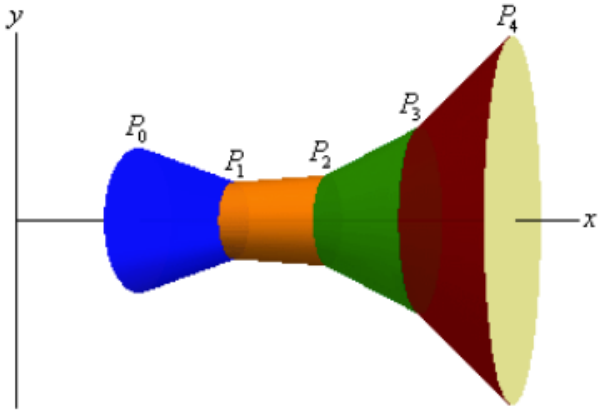
Area of Surface of Revolution

- The approximation on each interval gives a distinct portion of the solid and to make this clear, each portion is colored differently. Each of these portions are called **frustums** and we know how to find the surface area of **frustums**.



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Area of Surface of Revolution

The surface area of a frustum is given by

$$A = 2\pi r l$$

where

$$r = \frac{1}{2}(r_1 + r_2)$$

r_1 = radius of right end; r_2 = radus of left end;

and l is the length of the slant of the frustum.

So, for the frustum on the interval $[x_{i-1}, x_i]$ we have,

$$r_1 = f(x_i)$$
$$r_2 = f(x_{i-1})$$

$l = |P_{i-1} \quad P_i|$ (length of the line segment connecting P_i and P_{i-1})

Area of Surface of Revolution

and we know from the previous section that,

$$|P_{i-1}\ \ P_i| = \sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\ \Delta x$$

where x_i^* is some point in $[x_{i-1}, x_i]$.

Assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

So, the surface area of the frustum on the interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)|P_{i-1}\ \ P_i|$$

$$\approx 2\pi f(x_i^*)\sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\,\Delta x$$

Area of Surface of Revolution

The surface area of the whole solid is then approximately

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + \left[f'\left(x_i^*\right)\right]^2}\ \Delta x$$

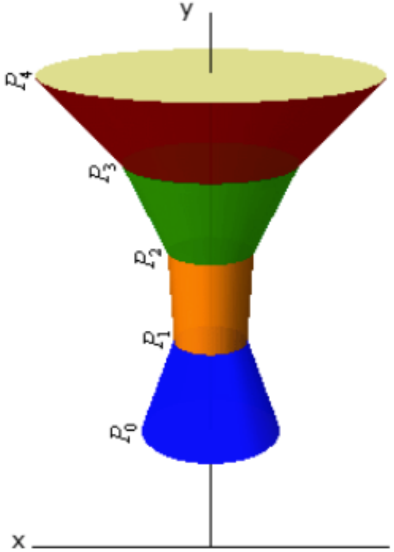
and we can get the exact surface area by taking the limit as n goes to infinity.

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + \left[f'\left(x_i^*\right)\right]^2}\ \Delta x$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ \ dx$$

Area of Surface of Revolution

We could also derive a similar formula for rotating $x = h(y)$ on $[c, d]$ around the y-axis.



This would give the following formula.

$$S = \int_c^d 2\pi h(y)\sqrt{1 + [h'(y)]^2}\ dy$$

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Summary

Surface Area Formulas given by

$$S = \int_a^b 2\pi y \, ds \qquad \text{(rotation about } x\text{-axis)}$$

$$S = \int_c^d 2\pi x \, ds \qquad \text{(rotation about } y\text{-axis)}$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx \quad \text{if } y = f(x), a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \; dx \quad \text{if x} = h(y), c \le y \le d$$

Example 1

Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution:

The formula that we'll be using here is

$$S = \int_{-2}^{2} 2\pi y \, ds$$

Since we are rotating about the x-axis and we'll use the first ds in this case because our function is in the correct form for that ds

Let's first get the derivative and the root taken care of.

$$\frac{dy}{dx} = \frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{(9 - x^2)^{-\frac{1}{2}}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

Example 1

Solution (continuation):

Here's the integral for the surface area

$$S = \int_{-2}^{2} 2\pi y \frac{3}{\sqrt{9 - x^2}}\, dx$$

There is a problem however. The dx means that we shouldn't have any y's in the integral. So, before evaluating the integral we'll need to substitute in for y as well.

The surface area is then,

$$S = \int_{-2}^{2} 2\pi\sqrt{9 - x^2}\,\frac{3}{\sqrt{9-x^2}}\, dx$$

$$= \int_{-2}^{2} 6\pi\ dx$$

$$= 24\pi$$

Example 2

Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the y-axis.

Solution:

Note that we've been given the function set up for the first ds, and the limits that work for the second ds.

Solution 1

This solution will use the first ds. We'll start with the derivative and root.

$$\frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{9x^{\frac{4}{3}}}} = \sqrt{\frac{9x^{\frac{4}{3}} + 1}{9x^{\frac{4}{3}}}} = \frac{\sqrt{9x^{\frac{4}{3}} + 1}}{3x^{\frac{2}{3}}}$$

Example 2

Solution (continuation):

We'll also need to get the limits. That isn't too bad however. All we need to do is plug in the given y's into our equation and solve to get that the range of x's is $1 \leq x \leq 8$. The integral for the surface area is then

$$S = \int_1^8 2\pi x \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}\, dx$$

$$= \frac{2\pi}{3}\int_1^8 x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}}+1}\, dx$$

(continue this by subtitution method)

Example 2

Solution (continuation):

Solution 2

This solution we'll use the second ds. So, we'll need to solve the equation for x. We'll also go ahead and get the derivative and root while we're at it.

$$x = y^3 \qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9y^4}$$

The surface area us then,

$$S = \int_1^2 2\pi x \sqrt{1 + 9y^4}\, dy$$

Exercise

Determine the surface area of the solid obtained by rotating $y = \sqrt{1 - x^2}$ around the x-axis, with $-1 \leq x \leq 1$.

Thank You