Applications of Integration: Area

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Area

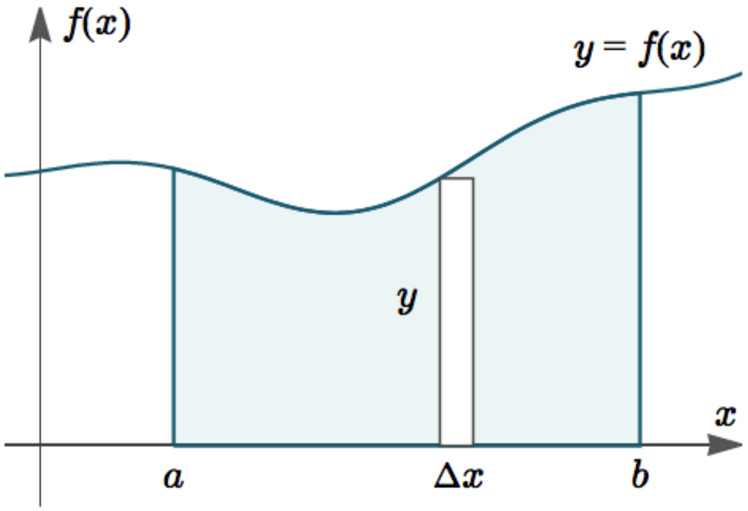
1. Area within Cartesian Coordinate System
2. Area within Polar Coordinate System

Area within Cartesian Coordinate System

We wish to find the area under the curve $y = f(x)$ from $x = a$ to $x = b$. We can have several situations.

- Curves which are entirely above the x-axis
- Curves which are entirely below the x-axis
- Part of the curve is below the x-axis, part of it is above the x-axis
- Certain curves are much easier to sum vertically

Curves which are entirely above the x-axis



The curve $y = f(x)$, completely above the x-axis. Shows a "typical" rectangle, Δx wide and y high.

(source: https://intmath.com)

In the picture above, a “typical rectangle” is shown with width Δx and high y. Its area is $\Delta x \cdot y$.

Curves which are entirely above the x-axis

If we add all these typical rectangles, starting from a and finishing at b, the area is approximately:

$$\sum_{x=a}^{b}(y)\Delta x$$

Now, if we let $\Delta x \to 0$, we can find the exact area by integration:

$$Area = \int_{a}^{b} f(x)dx$$

This follows from the Riemann Sums, from the introduction to integration chapter.

Example 1.

Find the area below $f(x) = 9 - x^2$ over the interval $-2 \leq x \leq 2$.

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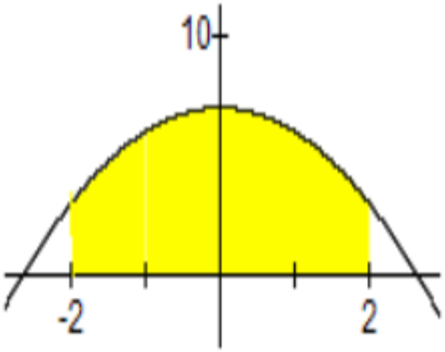
Solution:



Hence, the area of the picture above will be:

$$L = \int_{-2}^{2} (9 - x^2) dx$$

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Example 2.

Find the area underneath the curve $y = x^2 + 2$ from $x = 1$ to $x = 2$.

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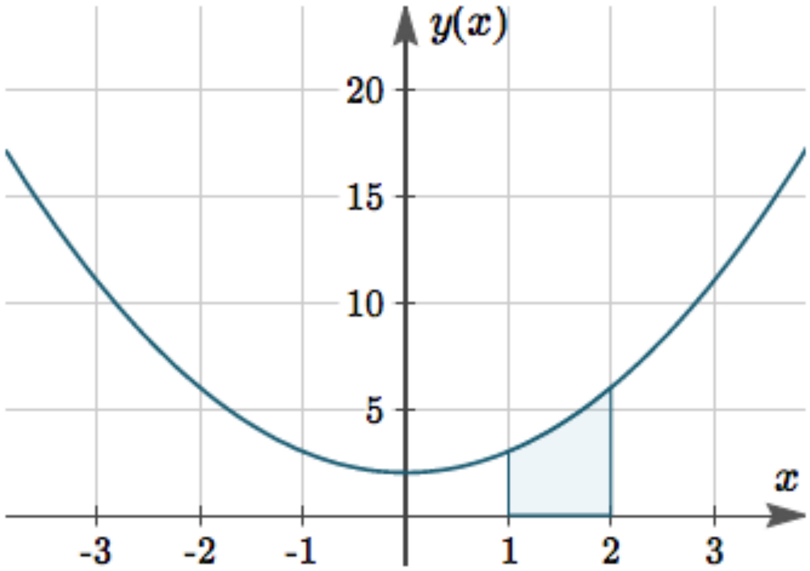

(source: https://intmath.com)

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Example 2.

Find the area underneath the curve $y = x^2 + 2$ from $x = 1$ to $x = 2$.

Solution:

$$Area = \int_1^2 (x^2 + 2)\, dx$$

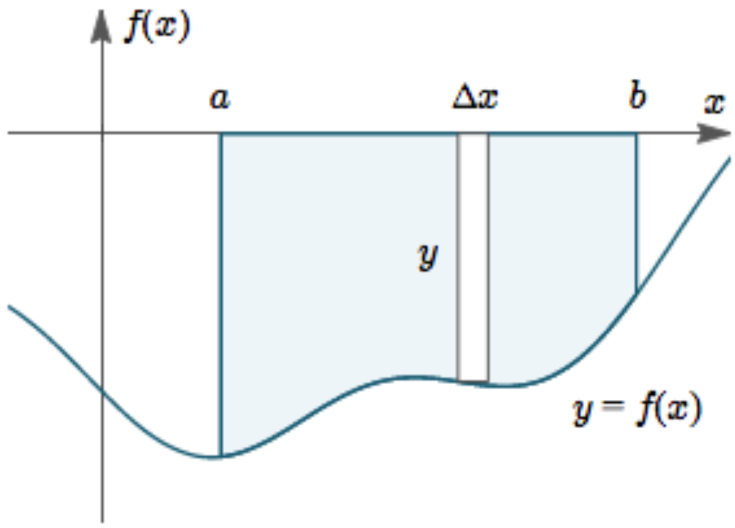
$$= \left[\frac{x^3}{3} + 2x\right]_1^2$$

$$= \left[\left(\frac{8}{3} + 4\right) - \left(\frac{1}{3} + 2\right)\right]$$

$$= \frac{13}{3} \qquad \text{units}^2$$

Curves which are entirely below the x-axis

We consider the case where the curve is below the x-axis for the range of x values being considered.



The curve $y = f(x)$, completely below the x-axis. Shows a "typical" rectangle, Δx wide and y high.

(source: https://intmath.com)

Curves which are entirely below the x-axis

In this case, the integral gives a **negative number**. We need to take the **absolute value** of this to find out area:

$$Area = \left| \int_a^b f(x)\, dx \right|$$

Example

Find the area above the curve $y = x^2 - 4$ over the interval $-1 \leq x \leq 2$.

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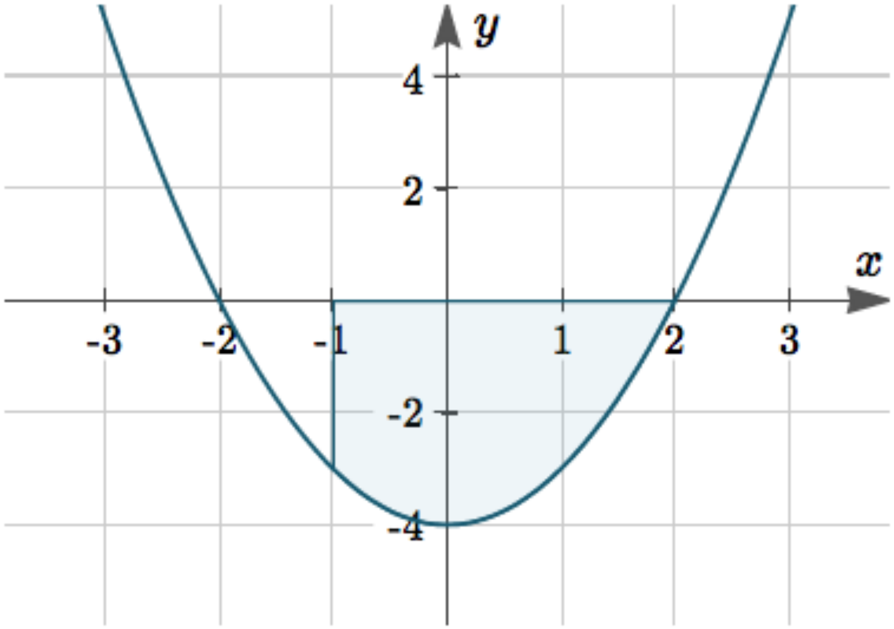


(source: https://intmath.com)

Example

Find the area above the curve $y = x^2 - 4$ over the interval $-1 \leq x \leq 2$.

Solution:

$$Area = \left|\int_{-1}^{2} (x^2 - 4)\, dx\right|$$

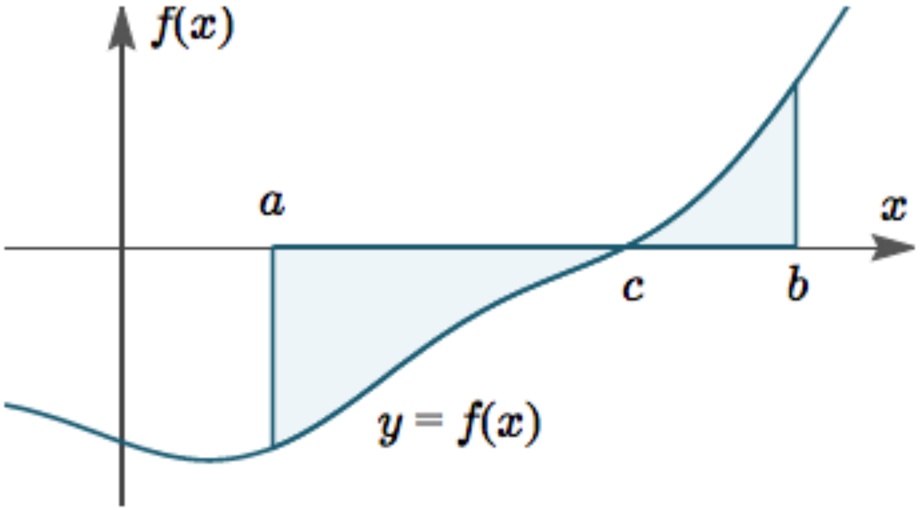
$$= \left|\left[\frac{x^3}{3} - 4x\right]_{-1}^{2}\right|$$

$$= \left|\left[\left(\frac{8}{3} - 8\right) - \left(-\frac{1}{3} + 4\right)\right]\right|$$

$$= |-9|$$

$$= 9 \qquad \text{units}^2$$

Part of the curve is below the x-axis, part of it is above the x-axis



From a to c, the curve $y = f(x)$ is below the x-axis, and from c to b, it is above.
(source: https://intmath.com)

In this case, we have to sum the individual parts, taking the absolute value for the section where the curve is below the x-axis (from $x = a$ to $x = c$).

Part of the curve is below the x-axis, part of it is above the x-axis

$$Area = \left|\int_a^c f(x)dx\right| + \int_c^b f(x)dx$$

If we don’t do it (the absolute sign) like this, the “negative” area (the part below the x-axis) will be substracted from the “positive” part, and our total area will not be correct.

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Example

What is the area bounded by the curve $y = x^2 - 4$, over the interval $0 \leq x \leq 4$?

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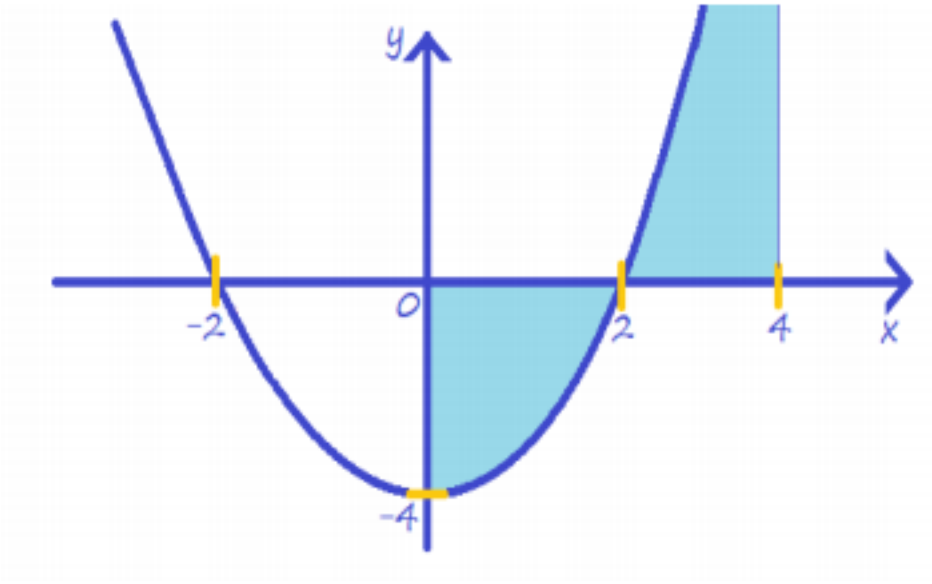


Example

What is the area bounded by the curve $y = x^2 - 4$, over the interval $0 \leq x \leq 4$?

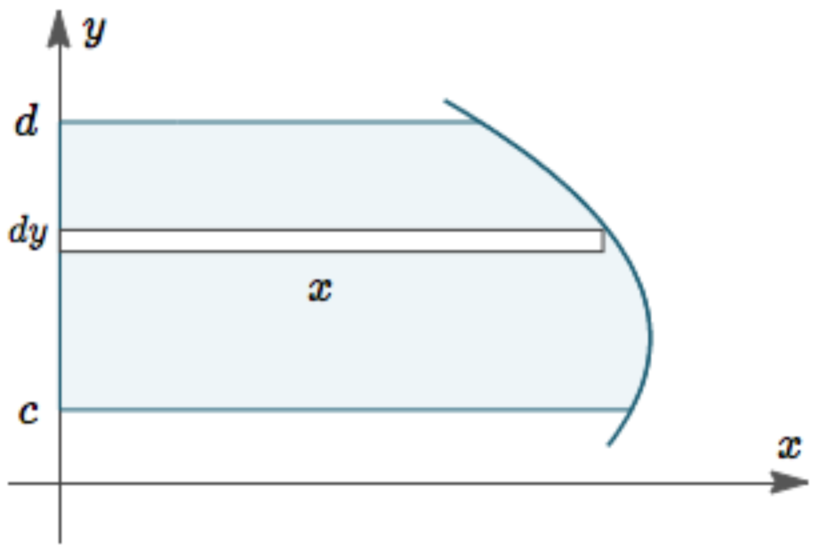
Solution:

Thea area can be calculated by this way.

$$L = |L_1| + L_2$$
$$= \left|\int_0^2 (x^2 - 4)\, dx\right| + \int_2^4 (x^2 - 4)\, dx$$

Certain curves are much easier to sum vertically

In some cases, it is easier to find the area if we take **vertical** sums. Sometimes the only possible way is to sum vertically.



The best way to find the area under this curve is by summing vertically.

(source: https://intmath.com)

Certain curves are much easier to sum vertically

In this case, we are trying to find the area of rectangles, with heights $x = f(y)$ and width dy.

If we are given $y = f(x)$ then we need to re-express this as $x = f(y)$ and we need to **sum from bottom to top**. So,

$$Area = \int_c^d f(y)dy$$

Example

Find the are of the region bounded by the curve $y = \sqrt{x-1}$, the y-axis and the lines $y = 1$ and $y = 5$.

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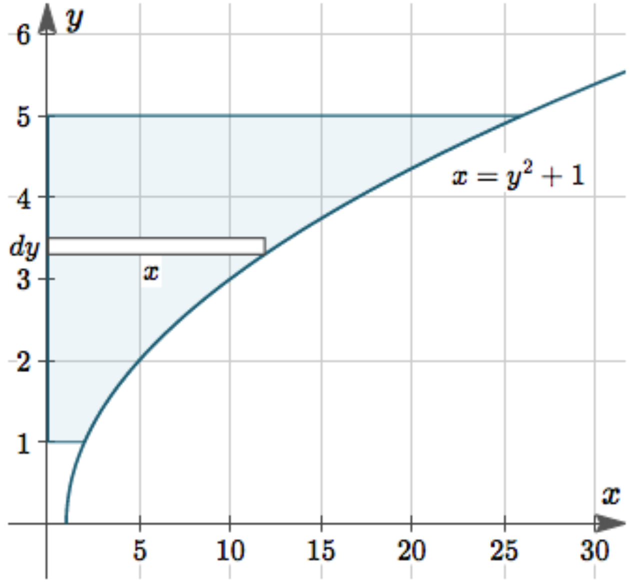
Solution:



(source: https://intmath.com)

Example

Find the are of the region bounded by the curve $y = \sqrt{x-1}$, the y-axis and the lines $y = 1$ and $y = 5$.

Solution: In this case, we express x as a function of y:

$$y = \sqrt{x-1}$$

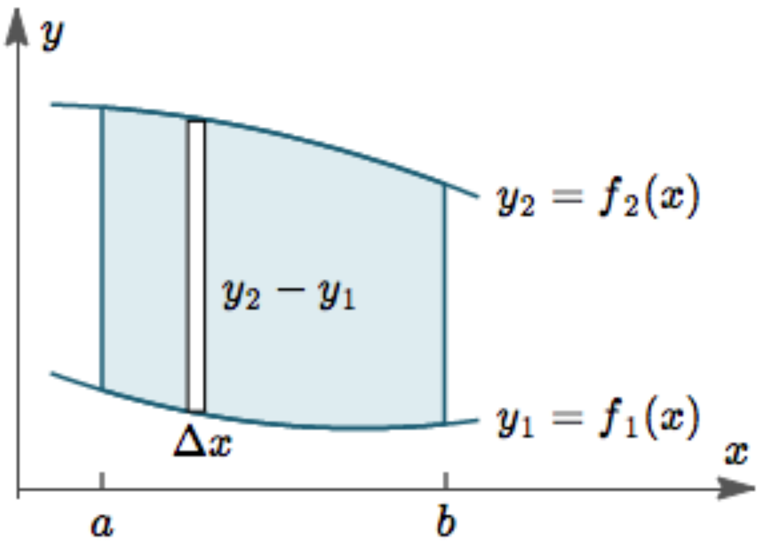
$$y^2 = x - 1$$

$$x = y^2 + 1$$

So, the area is given by:

$$Area = \int_1^5 (y^2 + 1)\, dy = \left[\frac{y^3}{3} + y\right]_1^5 = 45\frac{1}{3} \qquad \text{units}^2$$

Area between 2 curves using integration



Area bounded by the curves y_1 and y_2, & the lines $x = a$ and $x = b$, including a typical rectangle.

(source: https://intmath.com)

We are trying to find the area between 2 curves, $y_1 = f_1(x)$ and $y_2 = f_2(x)$, and the lines $x = a$ and $x = b$.

Area between 2 curves using integration

Based on the picture, we see if we substract the area under the lower curve

$$y_1 = f_1(x)$$

from the area under the upper curve

$$y_2 = f_2(x)$$

then we will find the required area. This can be done in one step:

$$Area = \int_a^b (y_2 - y_1)dx$$

Alternative way to find the formula (from first principles)

Another way of deriving this formula is as follows:

Each "typical rectangle" indicated has width Δx and height $y_2 - y_1$, so its area is $(y_2 - y_1) \cdot \Delta x$.

If we add all these typical rectangles, starting from a to b, the area is approximately:

$$\sum_{x=a}^{b} (y_2 - y_1)\Delta x$$

Now if we let $\Delta x \to 0$, we can find the exact area by integration:

$$Area = \int_a^b (y_2 - y_1)dx$$

Summing vertically to find area between 2 curves

Likewise, we can sum vertically by re-expressing both functions so that they are functions of y and we find:

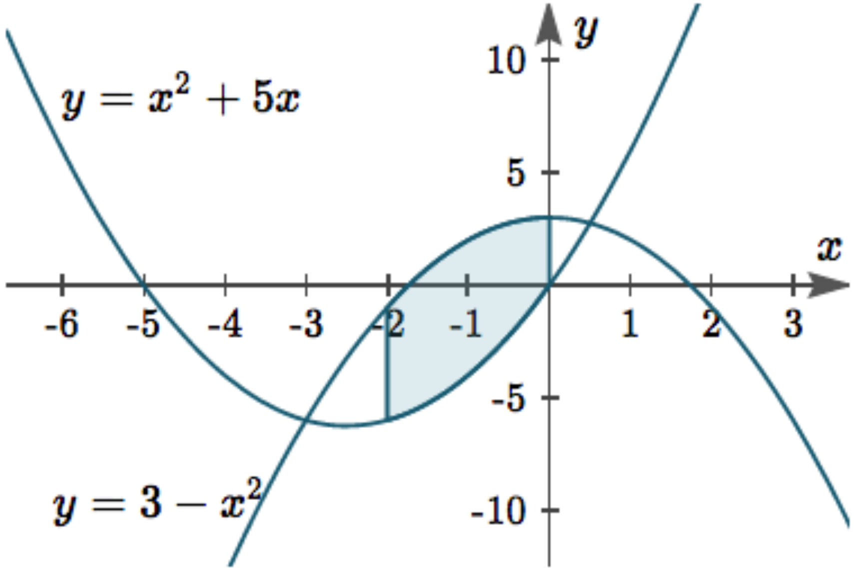
$$Area = \int_{c}^{d} (x_2 - x_1) dy$$

Notice the c and d as the limits on the integral (to remind us we are summing vertically) and the dy. It reminds us to express our function in terms of y.

Example

Find the area between the curves $y = x^2 + 5x$ and $y = 3 - x^2$ between $x = -2$ and $x = 0$.

Solution:



(source: https://intmath.com)

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Example

Find the area between the curves $y = x^2 + 5x$ and $y = 3 - x^2$ between $x = -2$ and $x = 0$.

Solution: From the graph, we see that $y = 3 - x^2$ is above $y = x^2 + 5x$ in the region of interest, so we will use:

$y_2 = 3 - x^2$, and

$y_1 = x^2 + 5x$

Then, we need to calculate:

$$Area = \int_a^b (y_2 - y_1)\, dx = \int_{-2}^{0} [(3 - x^2) - (x^2 + 5x)]\, dx$$

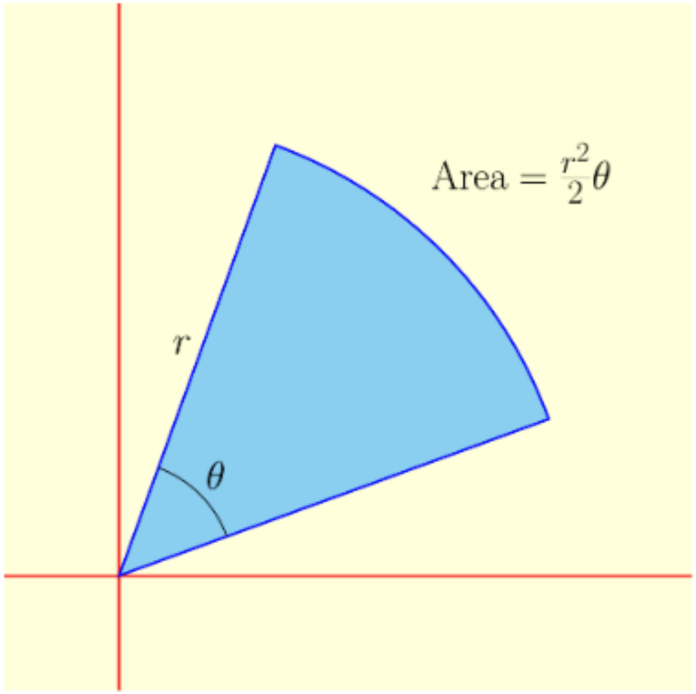
Exercises

1. Find the area bounded by $y = x^3$, $x = 0$ and $y = 3$.
2. Find the area bounded by the curves $y = x^2 + 5x$ and $y = 3 - x^2$.
3. Find the area bounded by the curves $y = x^2$, $y = 2 - x$, and $y = 1$.

Area within Polar Coordinate System

To understand the area inside of a polar curve $r = f(\theta)$, we start with the area of a slice of pie. If the slice has angle θ and radius r, then it is a fraction $\frac{\theta}{2\pi}$ of the entire pie.

So its area is: $\frac{\theta}{2\pi}\pi r^2 = \frac{r^2}{2}\theta$

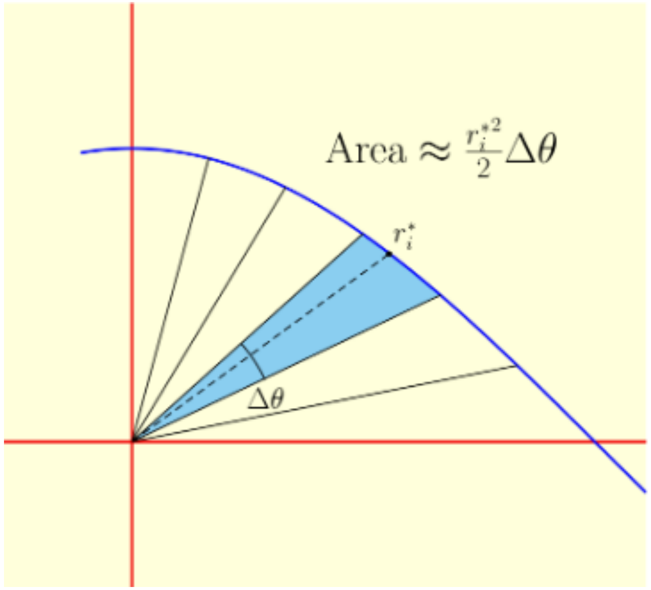


(source: https://web.ma.utexas.edu)

Area within Polar Coordinate System

Now we can compute the area inside of polar curve $r = f(\theta)$ between angles $\theta = \alpha$ and $\theta = \beta$. As with all bulk quantities, we

1. Break the region into N small pieces
2. Estimate the contribution of each piece.
3. Add up the pieces.
4. Take a limit to get an integral.



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Area within Polar Coordinate System

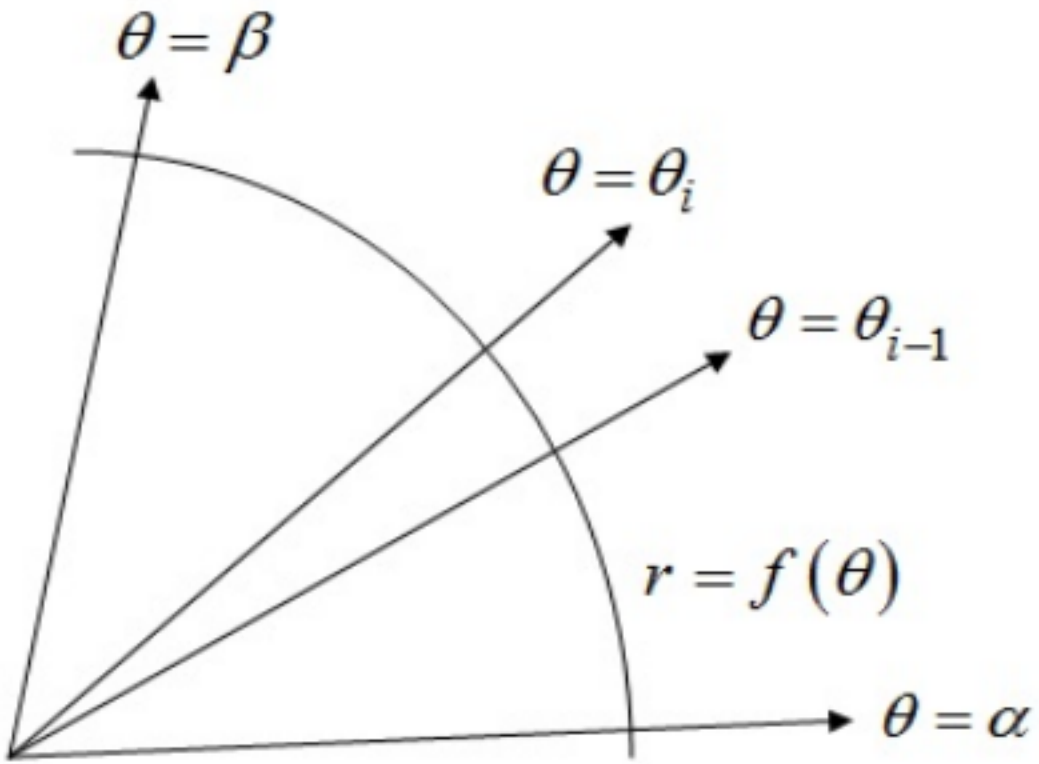
In our case, the pieces of angle $\Delta\theta = {}^{(\beta-\alpha)}/_{N}$. These aren't exactly pie slices, since the radius isn't constant, but it's a good approximation when N is large and $\Delta\theta$ is small. The i-th slice has area approximately $f(\theta_i^*)^2\,{}^{\Delta\theta}/_{2}$, where θ_i^* is a representative angle between α $+ (i-1)\Delta\theta$ and α $+ i\Delta\theta$, so the whole thing has are approximately $\sum_{i=1}^{N} \frac{f(\theta_i^*)^2}{2}\Delta\theta$. Taking a limit as $N \to \infty$ gives the integral

$$\int_{\alpha}^{\beta} \frac{f(\theta)^2}{2}\,d\theta$$

or equivalently

$$\frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta$$

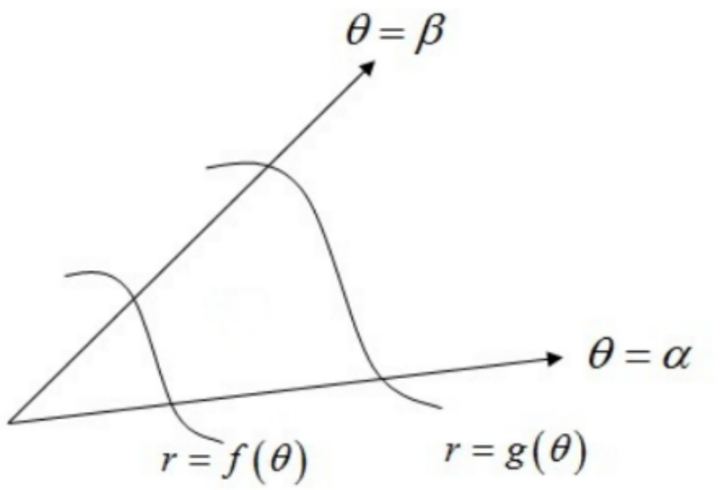
Illustration



Area within Polar Coordinate System

Furthermore, it is easy to understand that if

$$A = \{(r, \theta) | f(\theta) \leq r \leq g(\theta), \alpha \leq \theta \leq \beta\}$$



then the area of A is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (g(\theta))^2 d\theta - \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \left\{(g(\theta))^2 - (f(\theta))^2\right\} d\theta$$

Example 1.

Given $r_1 = \cos\theta$ and $r_2 = 1 - \cos\theta$. Find the area inside r_1 and outside r_2.

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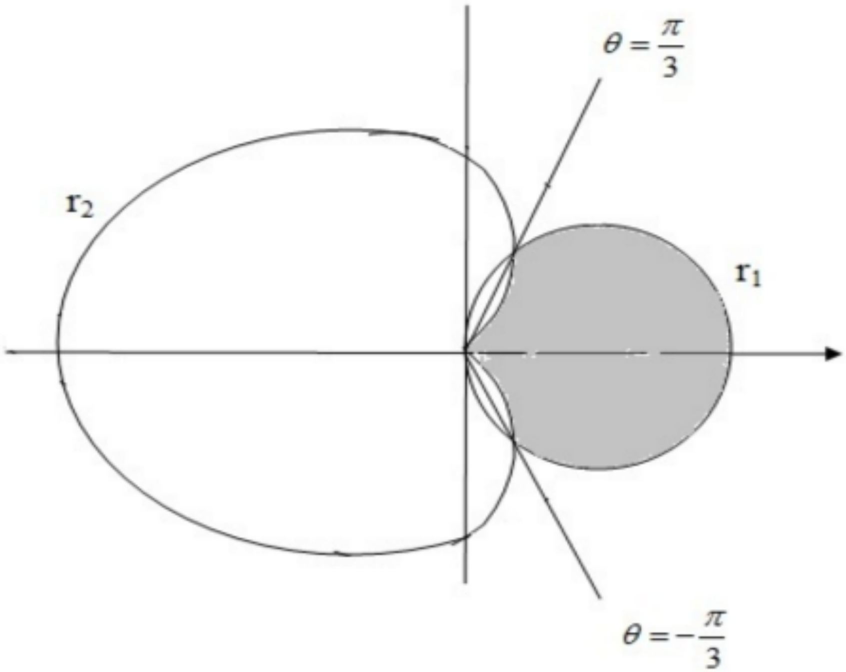
Given $r_1 = \cos\theta$ and $r_2 = 1 - \cos\theta$. Find the area inside r_1 and outside r_2.

Solution: let's look at the following picture



Example 1.

Firstly, find the intersection of both curves, which is:

$$\cos\theta = 1 - \cos\theta$$

$$\Leftrightarrow 2\cos\theta = 1$$

$$\Leftrightarrow \cos\theta = \frac{1}{2}$$

$$\Leftrightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

Look at the area on the 1st quadrant. It is the area where the area of r_1 substracted by the area of r_2, θ start from 0 to $\frac{\pi}{3}$. Furthermore, we need to double that area.

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Example 1.

So the area will be

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} (r_1^2 - r_2^2)\, d\theta\right) = \int_0^{\frac{\pi}{3}} (\cos^2\theta - (1-\cos\theta)^2)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} (\cos^2\theta - 1 + 2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} (2\cos\theta - 1)\, d\theta = [2\sin\theta - \theta]_0^{\frac{\pi}{3}}$$

$$= \sqrt{3} - \frac{\pi}{3}$$

Example 2.

Given $r_1 = \cos\theta$ and $r_2 = 1 - \cos\theta$. Find the area inside the r_1 and r_2.

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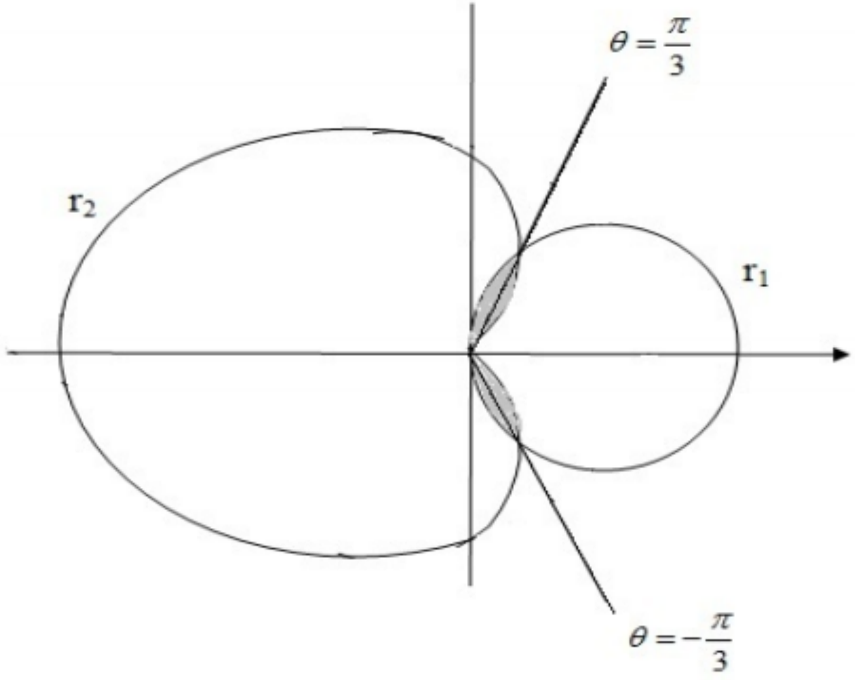

Example 2.

So, the area will be:

$$A = 2\left(\frac{1}{2}\int_{0}^{\frac{\pi}{3}}(r_2^2)\,d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(r_1^2)\,d\theta\right)$$

$$= \int_{0}^{\frac{\pi}{3}}(1-\cos\theta)^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(\cos\theta)^2 d\theta$$

$$= \int_{0}^{\frac{\pi}{3}}(1-2\cos\theta+\cos^2\theta)\,d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$= \int_{0}^{\frac{\pi}{3}}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{0}^{\frac{\pi}{3}} + \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - \sqrt{3} + \frac{1}{4}\frac{1}{2}\sqrt{3}\right) + \left(\frac{\pi}{4} + 0 - \frac{\pi}{6} - \frac{1}{4}\frac{1}{2}\sqrt{3}\right) = \frac{7\pi}{12} - \sqrt{3}$$

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Exercises

Find the area:

1) Inside $r = \sin\theta$ and $r = \cos\theta$
2) Inside $r = 2\sin\theta$ and $r = 1 - \cos\theta$
3) Inside $r = 2 + \sin\theta$ and $r = 5\sin\theta$
4) Inside $r = 3\sin\theta$ and $r = 2 + \cos\theta$
5) Inside $r = 3\cos 2\theta$
6) Inside $r = 1$ and $r = \cos\theta$
7) Inside the cardiode $r = 1 + \cos\theta$
8) Inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$
9) Outside $r = 1 - \cos\theta$ and inside $r = 2\sin\theta$
10) Inside $r = 2$ and outside $r = 1 - \cos\theta$

Thank You