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Applications of Integration: Center of Mass in Polar System

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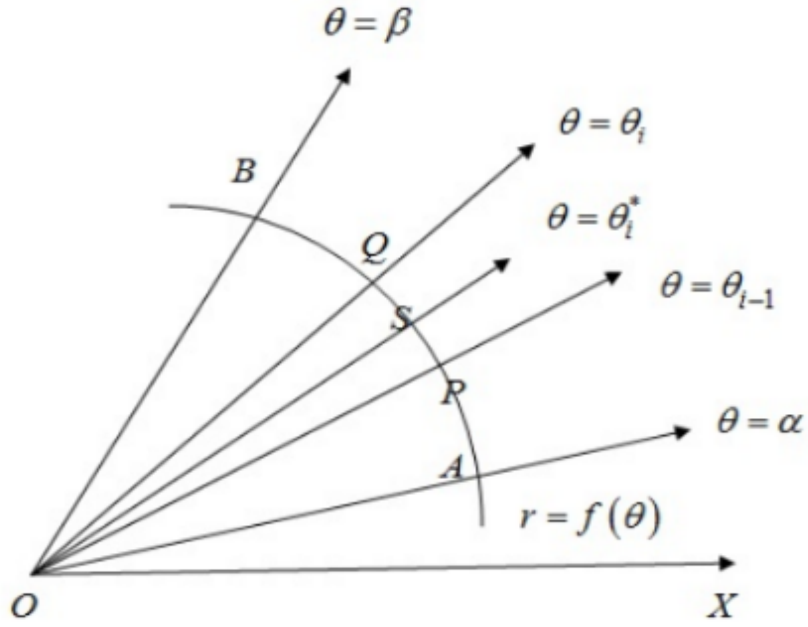
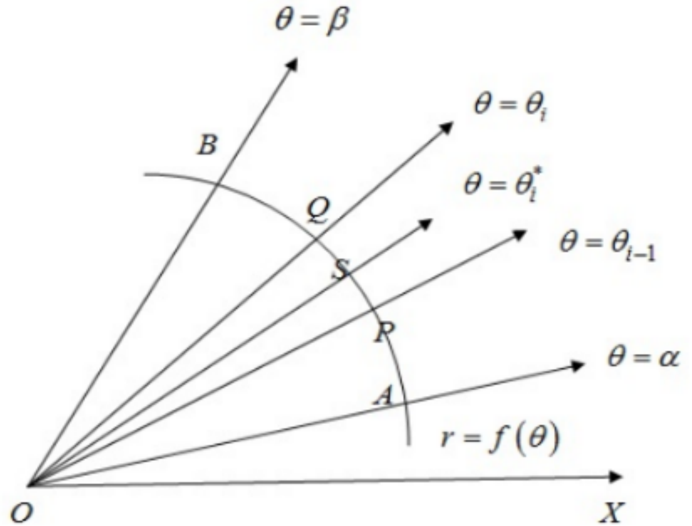
Center of Mass in Polar Coordinate System

- Given an area OAB bounded by curves $r = f(\theta)$, lines $\theta = \alpha$, and $\theta = \beta$. We are going to find the center of mass of OAB.



Center of Mass in Polar Coordinate System



- We form partitions $p = \{\alpha = \theta_0, \theta_1, \theta_2, \dots, \theta_n = \beta\}$ on $[\alpha, \beta]$. For each $i = 1,2, \dots, n$, we have $\theta_i^* = \frac{\theta_{i-1}+\theta_i}{2} \in [\theta_{i-1}, \theta_i]$ and $\Delta\theta_i = \theta_i - \theta_{i-1}$. Next, we approach the area of OPQ by isosceles triangle with high $OS = r_i = f(\theta_i^*)$. The center of mass coordinate of this isosceles triangle will be $\left(\frac{2}{3} r_i, \theta_i^*\right)$.

Center of Mass in Polar Coordinate System

- By the definition of center of mass and the definite integral, the center of mass of area OAB is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{2} r_i^2 \Delta\theta_i\right)\left(\frac{2}{3} r_i \cos\theta_i^*\right)}{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta_i} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{3} r_i^3 (\cos\theta_i^*) \Delta\theta_i}{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta_i}$$

$$= \frac{\int_\alpha^\beta \frac{1}{3} r^3 \cos\theta \, d\theta}{\int_\alpha^\beta \frac{1}{2} r^2 d\theta} = \frac{\frac{2}{3}\int_\alpha^\beta r^3 \cos\theta \, d\theta}{\int_\alpha^\beta r^2 \, d\theta}$$

and,

$$\bar{y} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{2} r_i^2 \Delta\theta_i\right)\left(\frac{2}{3} r_i \sin\theta_i^*\right)}{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta_i} = \frac{\int_\alpha^\beta \frac{1}{3} r^3 \sin\theta \, d\theta}{\int_\alpha^\beta \frac{1}{2} r^2 d\theta} = \frac{\frac{2}{3}\int_\alpha^\beta r^3 \sin\theta \, d\theta}{\int_\alpha^\beta r^2 \, d\theta}$$

Summary: the Center of Mass in Polar System

The coordinate of center of mass in polar system is given by $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\frac{2}{3}\int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta}{\int_{\alpha}^{\beta} r^2 \, d\theta}$$

and

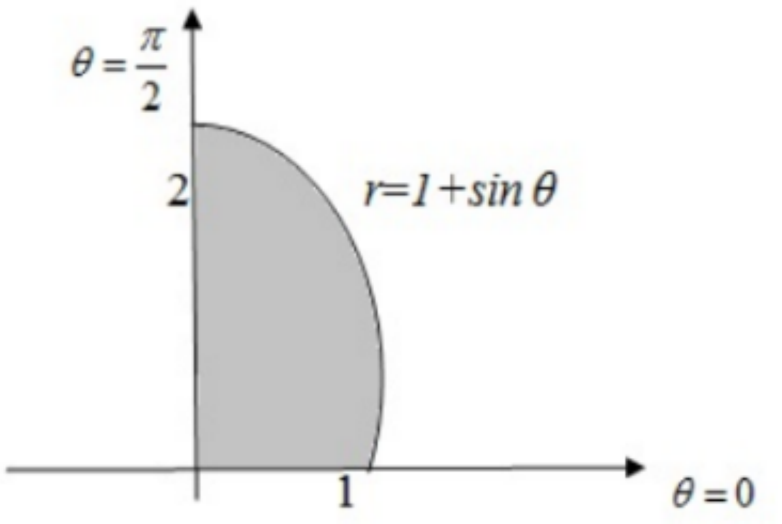
$$\bar{y} = \frac{\frac{2}{3}\int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta}{\int_{\alpha}^{\beta} r^2 \, d\theta}$$

Example 1

Determine the center of mass of the region bounded by $r = 1 + \sin\theta$, where $\theta \in \left[0, \frac{\pi}{2}\right]$.

Solution:

The sketch of the region given as folow



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Example 1

Solution (continuation):

The coordinate of the center of mass is

$$\bar{x} = \frac{\frac{2}{3}\int_0^{\frac{\pi}{2}} r^3 \cos\theta \, d\theta}{\int_0^{\frac{\pi}{2}} r^2 \, d\theta} = \frac{\frac{2}{3}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^3 \cos\theta \, d\theta}{\int_0^{\frac{\pi}{2}} (1+\sin\theta)^2 \, d\theta}$$

$$= \frac{\frac{2}{3}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^3 d(1+\sin\theta)}{\int_0^{\frac{\pi}{2}} (1+2\sin\theta+\sin^2\theta) \, d\theta}$$

$$= \frac{\frac{2}{3}\left[\frac{1}{4}(1+\sin\theta)^4\right]_0^{\frac{\pi}{2}}}{\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}} = \frac{\frac{2}{3}\left(4-\frac{1}{4}\right)}{\frac{3\pi}{4}} = \frac{10}{3\pi}$$

Example 1

Solution (continuation):

and,

$$\bar{y} = \frac{\frac{2}{3}\int_0^{\frac{\pi}{2}} r^3 \sin\theta \, d\theta}{\int_0^{\frac{\pi}{2}} r^2 \, d\theta} = \frac{\frac{2}{3}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^3 \sin\theta \, d\theta}{\int_0^{\frac{\pi}{2}} (1+\sin\theta)^2 \, d\theta}$$

$$= \frac{\frac{2}{3}\int_0^{\frac{\pi}{2}} \left(\sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta\right) d\theta}{\int_0^{\frac{\pi}{2}} (1 + 2\sin\theta + \sin^2\theta) \, d\theta}$$

$$= \frac{\frac{2}{3}\left[\frac{15x}{8} - \sin 2\theta + \frac{1}{32}\sin 4\theta - \frac{13}{4}\cos\theta + \frac{1}{4}\cos 3\theta\right]_0^{\frac{\pi}{2}}}{\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}} = \frac{\frac{2}{3}\left(3 + \frac{15\pi}{16}\right)}{3\pi} = \frac{8}{3\pi} + \frac{5}{6}$$

Exercises

Determine the coordinate of center of mass of areas (in polar system) bellow:

a) Inside the curve $r = 2(1 - \cos\theta)$.

b) Inside the curve $r = 3 + 2\cos\theta$.

c) Inside the curve $r = 3\sin\theta$ and outside the curve $r = 1 + \sin\theta$.

d) Inside the curve $r = 2 + \sin\theta$ and $r = 5\sin\theta$.

Thank You