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Applications of Integration: Center of Mass of Solid Revolution

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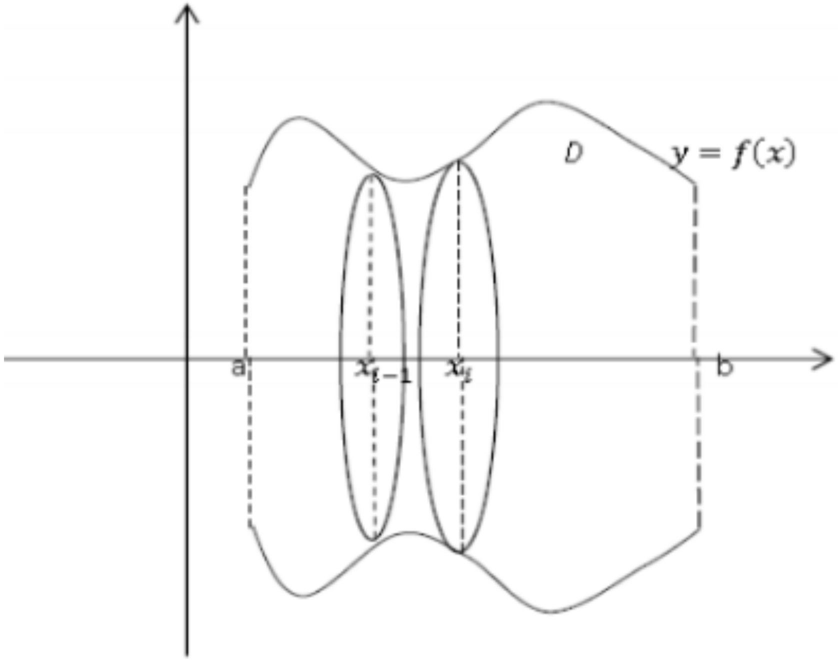
Center of Mass of Solid Revolution

Given f, a non-negative and integrable function on the interval $[a, b]$. So, we have the area $D = \{(x, y) |\ 0 \leq y \leq f(x), x \in [a, b]\}$ above the x-axis.



Center of Mass of Solid Revolution, where the function is $f(x)$

If D is revolved around the x-axis then we will have the homogeneous solid of revolution. Since the area is revolved around x-axis, the center of mass of that solid is located on x only, thus $y = 0$.

Suppose that ρ is the density constant. By the definition of center of mass discussed earlier, the coordinate of the solid of revolution will be $(\bar{x}, 0)$ where

$$\bar{x} = \frac{\int_a^b x\rho\pi\big(f(x)\big)^2 \, dx}{\int_a^b \rho\pi\big(f(x)\big)^2 \, dx} = \frac{\int_a^b x\big(f(x)\big)^2 \, dx}{\int_a^b \big(f(x)\big)^2 \, dx}$$

Center of Mass of Solid Revolution, where the function is $g(y)$

Similarly, if g is a non-negative and integrable function on the interval $[c, d]$ so that we have the area $D = \{(x, y) \mid 0 \leq x \leq g(y), y \in [c, d]\}$.

If D is revolved around the y-axis, then the center of mass of solid revolution is located on $(0, \bar{y})$ where

$$\bar{y} = \frac{\int_c^d y\rho\pi\left(g(y)\right)^2 dy}{\int_c^d \rho\pi\left(g(y)\right)^2 \, dy} = \frac{\int_c^d y\left(g(y)\right)^2 \, dy}{\int_c^d \left(g(y)\right)^2 \, dy}$$

Center of Mass of Solid Revolution, where functions are f_1 and f_2

Given a non-negative and integrable functions, f_1 and f_2, where $f_1(x) \geq f_2(x)$ for all $x \in [a, b]$. So we have the area $D = \{(x, y) \mid f_2(x) \leq y \leq f_1(x), x \in [a, b]\}$.

If the area D is revolved around x-axis, then the center of mass of that solid revolution will be $(\bar{x}, 0)$ where

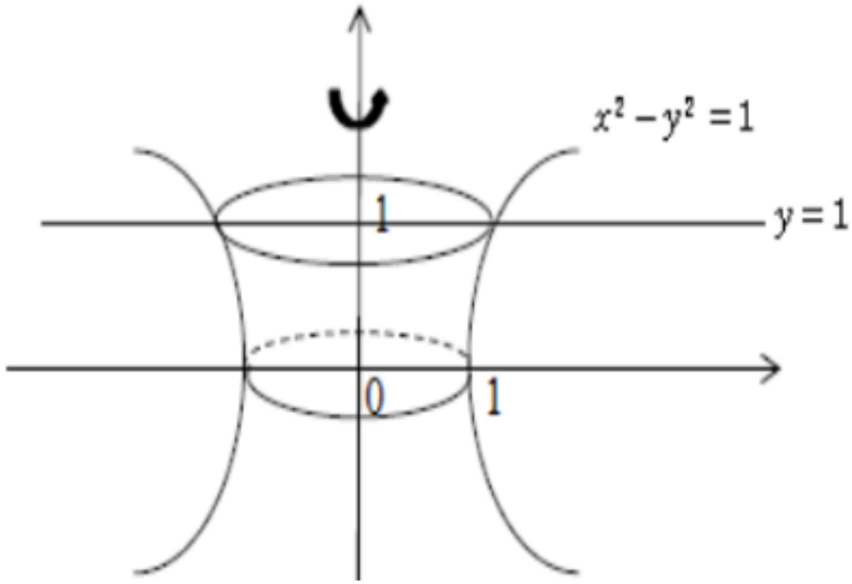
$$\bar{x} = \frac{\int_a^b x\left(\left(f_1(x)\right)^2 - \left(f_2(x)\right)^2\right) dx}{\int_a^b \left(\left(f_1(x)\right)^2 - \left(f_2(x)\right)^2\right) dx}$$

Example 1

Find the center of mass of solid revolution if the area that bounded by an hyperbolic

$x^2 - y^2 = 1$, x-axis, a straight line $x = 0$, and a straight line $y = 1$, revolved around the y-axis.

Solution:



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Example 1

Solution (continuation):

Suppose that the center of that solid revolution is $(\bar{x}, \bar{y})$, then $\bar{x} = 0$ and

$$\bar{y} = \frac{\int_0^1 y\left(1+y^2\right) dy}{\int_0^1 (1+y^2)\, dy}$$

$$= \frac{\int_0^1 \left(y+y^3\right) dy}{\int_0^1 (1+y^2)\, dy}$$

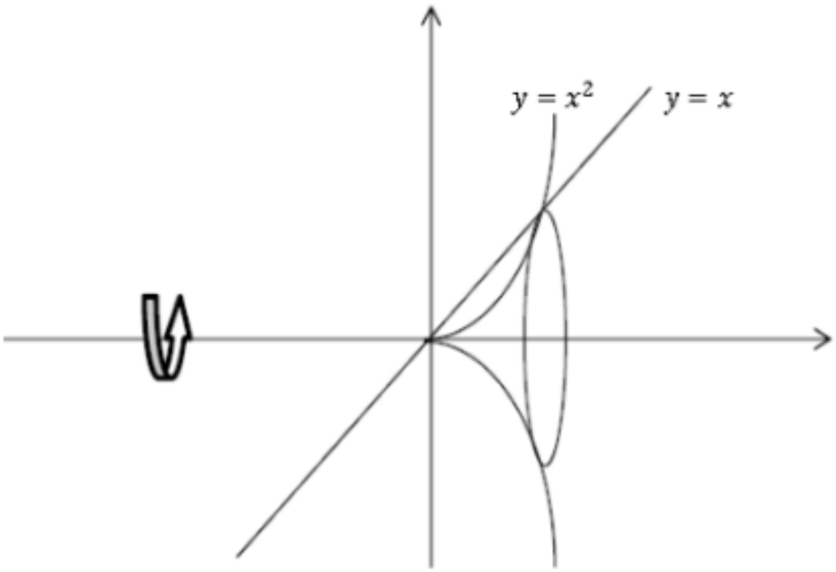
$$= \frac{\left[\frac{1}{2}y^2+\frac{1}{4}y^4\right]_0^1}{\left[y+\frac{1}{3}y^3\right]_0^1}$$

$$= \frac{\frac{1}{2}+\frac{1}{4}}{1+\frac{1}{3}} = \frac{9}{16}$$

Example 2

Find the center of mass of solid revolution if the area bounded by a parabolic $y = x^2$ and the straight line $y = x$, revolved around the x-axis.

Solution:



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Example 2

Solution (continuation):

Suppose that the center of mass of that solid revolution is $(\bar{x}, \bar{y})$, then

$\bar{y} = 0$ dan

$$\bar{x} = \frac{\int_0^1 x\left(x^2 - x^4\right)\,dx}{\int_0^1 (x^2 - x^4)\,dx}$$

$$= \frac{\int_0^1 \left(x^3 - x^5\right)\,dx}{\int_0^1 (x^2 - x^4)\,dx}$$

$$= \frac{\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1}{\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1}$$

$$= \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{3} - \frac{1}{5}} = \frac{5}{8}$$

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Exercise

1) Find the center of mass of solid revolution if the area bounded by a parabolic $y = 4x - x^2$ and a straight line $y = x$, revolved around the y-axis.
2) Find the center of mass of solid revolution if the area bounded by a parabolic $y = 4 - x^2$ in Ist Quadrant (Kuadran pertama), revolved around the x-axis.

Homework
(Submit before June 7th)

Center of mass of an area (Cartesian Coordinate)

1) Find the centroid of the region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$.

Center of mass of an area (Polar Coordinate)

2) Find the centroid of the region enclosed by the cardiode $r(\theta) = 1 + \cos\theta$.

Center of mass of an Arc

3) Determine the centroid (center of mass) of an Arc (busur) of $r = 2 + \sin\theta$, where θ start from 0 to $\frac{\pi}{2}$.

Center of mass of Surface Revolution

4) Determine the centroid of **surface revolution** if an arc $y = \frac{1}{2}x^2$, from $x = 0$ to $x = 1$, revolved around the y-axis.

Homework (Submit before June 7th)

Center of mass of Solid of Revolution

5) Find the center of mass of solid revolution if an area in I[st] Quadrant (Kuadran Pertama) bounded by x-axis, a parabolic $y = x^2$, and a straight line $y = -2x + 3$, revolved around:

   a) X-axis

   b) Y-axis

Thank You