UNIVERSITAS GADJAH MADA

Applications of Integration: Center of Mass

Ivan Luthfi Ihwani, M. Sc.

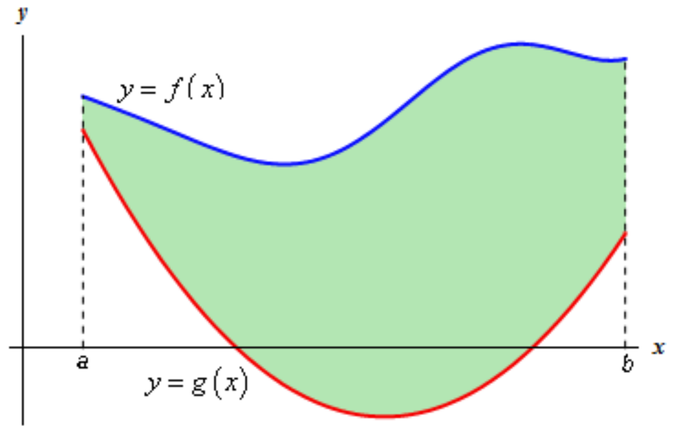
Center of Mass

- In this section, we are going to find the **center of mass** or **centroid** of a thin plate with uniform density ρ.
- The center of mass or centroid of a region is the point in which the region will be perfectly balanced horizontally if suspended from that point.

(source: *https://tutorial.math.lamar.edu*)

Center of Mass

- Let's suppose that the plate is the region bounded by the two curves $f(x)$ and $g(x)$ on the interval $[a, b]$. So, we want to find the center of mass of the region below.



(source: *https://tutorial.math.lamar.edu*)

Center of Mass

- We'll first need the mass of the plate. The mass is,

$$M = \rho \times (\text{Area of plate})$$
$$= \rho \int_a^b f(x) - g(x)\ dx$$

- Next, we'll need the **moments** of the region. There are two moments, denoted by M_x and M_y. The moments measure the tendency of the region to rotate about the x and y-axis respectively.

Equations of Moments

The moments are given by,

$$M_x = \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx$$

$$M_y = \rho \int_a^b x\big(f(x) - g(x)\big)\ dx$$

UNIVERSITAS GADJAH MADA

Center of Mass Coordinates

The coordinates of the center of mass $(\bar{x}, \bar{y})$, are then,

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x\big(f(x) - g(x)\big)\ dx}{\int_a^b f(x) - g(x)\ dx} = \frac{1}{A}\int_a^b x\big(f(x) - g(x)\big)\, dx$$

$$\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx}{\int_a^b f(x) - g(x)\ dx} = \frac{1}{A}\int_a^b \frac{1}{2}\ \left([f(x)]^2 - [g(x)]^2\right) dx$$

where,

$$A = \int_a^b f(x) - g(x)\ dx$$

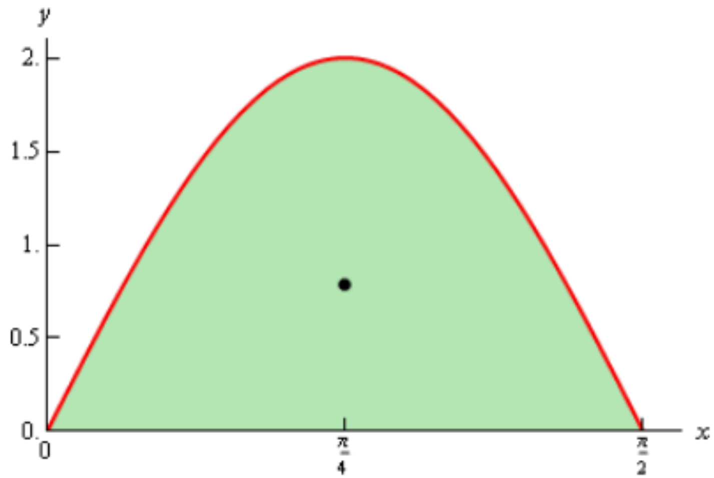
Note: *the density, ρ, of the plate cancels out and so it isn't really needed*

Example 1

Determine the center of mass for the region bounded by $y = 2\sin(2x)$, $y = 0$, on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution:

Here is a sketch of the region with the center of mass denoted with a dot.



(source: *https://tutorial.math.lamar.edu*)

Example 1

Solution (continuation):

Let's first get the area of the region.

$$A = \int_0^{\frac{\pi}{2}} 2\sin(2x)\,dx$$

$$= -\cos(2x)\Big|_0^{\frac{\pi}{2}}$$

$$= 2$$

UNIVERSITAS GADJAH MADA

Example 1

Solution (continuation):

Now, the moments (without density since it will just drop out) are,

$$M_x = \int_0^{\frac{\pi}{2}} 2\sin^2(2x)\ dx$$

$$= \int_0^{\frac{\pi}{2}} 1 - \cos(4x)\ dx$$

$$= \left(x - \frac{1}{4}\sin(4x)\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Example 1

Solution (continuation):

$$M_y = \int_0^{\frac{\pi}{2}} 2x \sin(2x)\ dx$$

$$= -x\cos(2x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(2x)\,dx$$

$$= -x\cos(2x)\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

The coordinates of the center of mass are then,

$$\bar{x} = \frac{\pi/2}{2} = \frac{\pi}{4} \qquad \text{and} \qquad \bar{y} = \frac{\pi/2}{2} = \frac{\pi}{4}$$

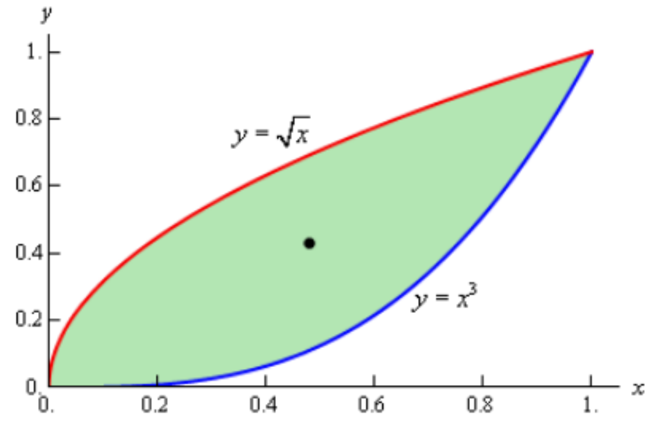
So, the center of mass for this region is $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Example 2

Determine the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$.

Solution:

The two curves intersect at $x = 0$ and $x = 1$ and here is a sketch of the region with the center of mass marked with a dot.



(source: *https://tutorial.math.lamar.edu*)

Example 2

UNIVERSITAS GADJAH MADA

Solution (continuation):

We'll first get the area of the region.

$$A = \int_0^1 \sqrt{x} - x^3 \; dx$$

$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^4\right)\Bigg|_0^1$$

$$= \frac{5}{12}$$

Example 2

Solution (continuation):

Now, the moments are

$$M_x = \int_0^1 \frac{1}{2}(x - x^6)\ dx$$
$$= \frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{7}x^7\right)\Big|_0^1$$
$$= \frac{5}{28}$$

Example 2

Solution (continuation):

Now, the moments are

$$M_y = \int_0^1 x(\sqrt{x} - x^3)\,dx$$

$$= \int_0^1 x^{\frac{3}{2}} - x^4 \; dx$$

$$= \left(\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{5}x^5\right)\Big|_0^1$$

$$= \frac{1}{5}$$

Example 2

Solution (continuation):

The coordinates of the center of mass is then,

$$\bar{x} = \frac{1/5}{5/12} = \frac{12}{25}$$

$$\bar{y} = \frac{5/28}{5/12} = \frac{3}{7}$$

The coordinates of the center of mass are then $\left(\frac{12}{25}, \frac{3}{7}\right)$.

Exercises

UNIVERSITAS GADJAH MADA

1) Determine the center of mass for the region bounded by $y = x$, $y = 0$, and $x = 3$.

2) Determine the center of mass for the region bounded by $y = \cos x$, $-\frac{\pi}{2} \leq x \leq -\frac{\pi}{2}$ and x-axis.

Thank You