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Applications of Integration: Volume of Solids of Revolution

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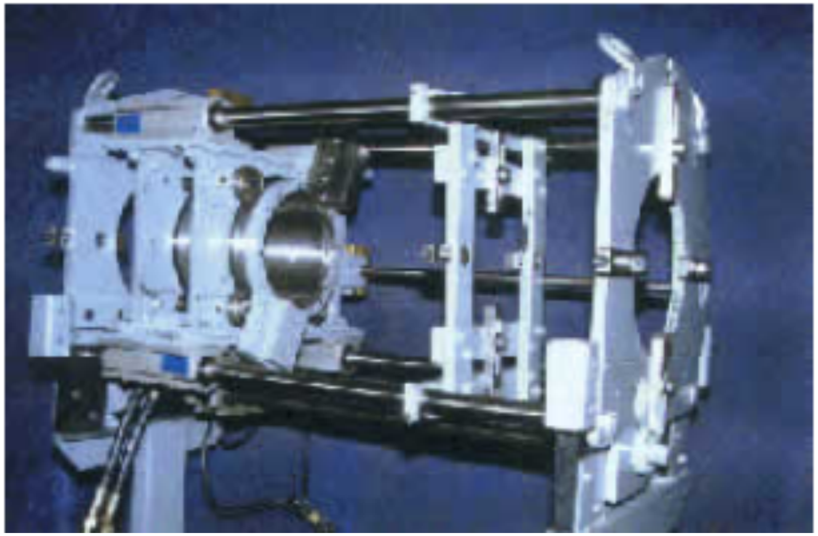
Volume of Solids of Revolution

1. Volume of Solids of Revolution on Cartesian Coordinate System
   a) Volume of Solids of Revolution by Disk Method
   b) Volume of Solids of Revolution by Shell Method (Method of Cylinders)
2. Volume of Solids of Revolution on Polar Coordinate System

Volume of Solids of Revolution

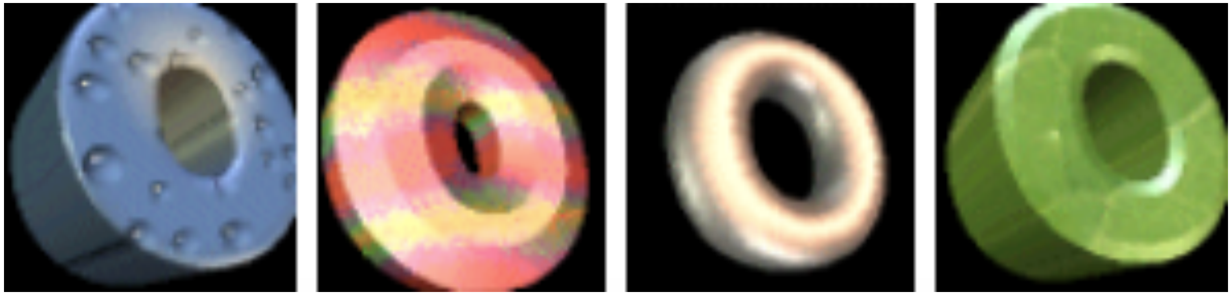
Many solid objects, especially those made on **a lathe** (ID: mesin bubut), have a circular cross-section and curved sides. On this section, we will see how to find the volume of such objects using integration.



A lathe
(source: *https://intmath.com*)

Volume of Solids of Revolution

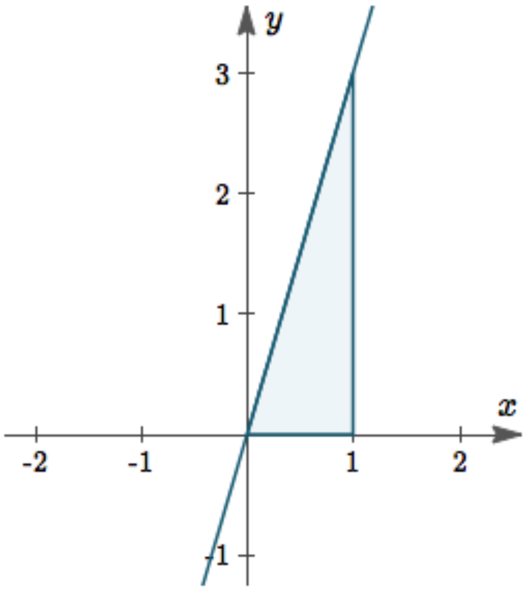
The following objects are made on a lathe.



Objects made on a lathe
(source: *https://intmath.com*)

Volume of Solids of Revolution on Cartesian Coordinate System (Example 1)

Consider the area bounded by straight line $y = 3x$, the x-axis, and $x = 1$:



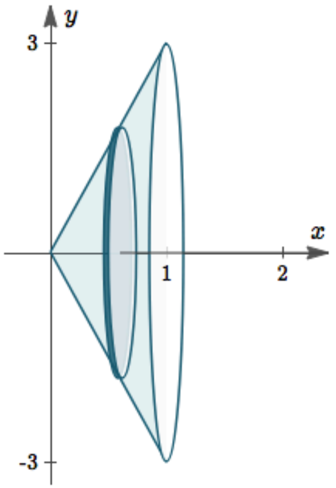
The graph of $y = 3x$, with the area under the "curve" between $x = 0$ to $x = 1$ shaded.

(source: *https://intmath.com*)

Volume of Solids of Revolution on Cartesian Coordinate System (Example 1)

When the shaded area is rotated 360^o about the x-axis, a volume is generated.

The resulting solid is a cone:

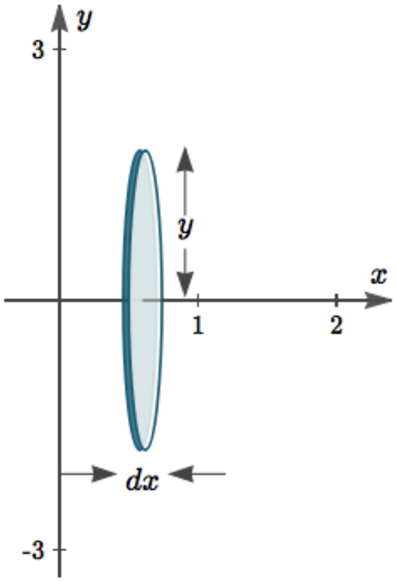


Area under the curve $y = 3x$ from $x = 0$ to $x = 1$ rotated around the x-axis, showing a typical disk.

(source: *https://intmath.com*)

Disk Method for Finding Volumes

To find a volume, we could take slices (the **dark green** disk shown on the previous picture is a typical slice), each dx wide and radius y:



The typical disk shown with its dimensions, radius = y and "height" = dx.

(source: *https://intmath.com*)

Disk Method for Finding Volumes

The **volume** of a cylinder is given by:

$$V = \pi r^2 h$$

Since the radius$= r = y$ and the high of each disk$= dx$, we notice that the volume of each slice is:

$$V = \pi y^2 dx$$

Adding the volumes of the disks (with infinitely small dx), we obtain the formula:

$V = \pi \int_a^b y^2\, dx$ or $V = \pi \int_a^b \{f(x)\}^2\, dx$

Disk Method for Finding Volumes

where:

- $y = f(x)$ is the equation of the curve whose area is being rotated,
- a and b are the limits of the area being rotated,
- dx shows that the area is being rotated about the x-axis.

Disk Method for Finding Volumes

Applying the formula $V = \pi \int_a^b y^2\, dx$ to the Example 1., we have:

$$
\begin{aligned}
Vol &= \pi \int_a^b y^2\, dx \\
&= \pi \int_0^1 (3x)^2\, dx \\
&= \pi \int_0^1 9x^2\, dx \\
&= \pi[3x^3]_0^1 \\
&= \pi[3] - \pi[0] \\
&= 3\pi \qquad \text{unit}^3 \qquad \text{(CHECK: USE THE CONE FORMULA!)}
\end{aligned}
$$

Exercises!

1) Find the volume if the area bounded by the curve $y = x^3 + 1$, the x-axis and the limits of $x = 0$ and $x = 3$ is rotated around the x-axis.
2) Find the volume if the area bounded by the curve $y = \sqrt{x}$, the x-axis, and the limits of $x = 0$ and $x = 4$ is rotated around the x-axis.
3) Find the volume if the area bounded by the straight line $y = x$, and $y = 0$, and $x = 2$, revolved about the x-axis.

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Washer Method: Volume by Rotating the Area Enclosed Between 2 Curves

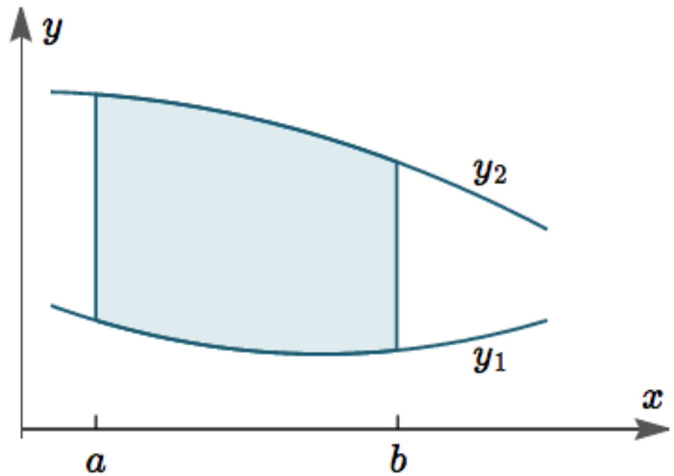
If we have 2 curves y_2 and y_1 that enclose some area and we rotate that area around the x-axis, then the volume of the solid formed is given by:

$$Volume = \pi \int_a^b [(y_2)^2 - (y_1)^2]\, dx$$

Notes: y_2 is above y_1.

Washer Method: Volume by Rotating the Area Enclosed Between 2 Curves

In the following general graph, y_2 is above y_1. The lower and upper limits for the region to be rotated are indicated by the vertical lines at $x = a$ and $x = b$.



Area bounded by the curves y_1 and y_2, & the lines $x = a$ and $x = b$.

(source: *https://intmath.com*)

Washer Method: Volume by Rotating the Area Enclosed Between 2 Curves

When we rotate such a shape around an axis, and take slices, the result is a **washer** shape (with a round hole in the middle).

Example

A cup-like object is made by rotating the area between $y = 2x^2$ and $y = x + 1$ with $x \geq 0$ around the x-axis. Find the volume of the material needed to make the cup. (units are cm)

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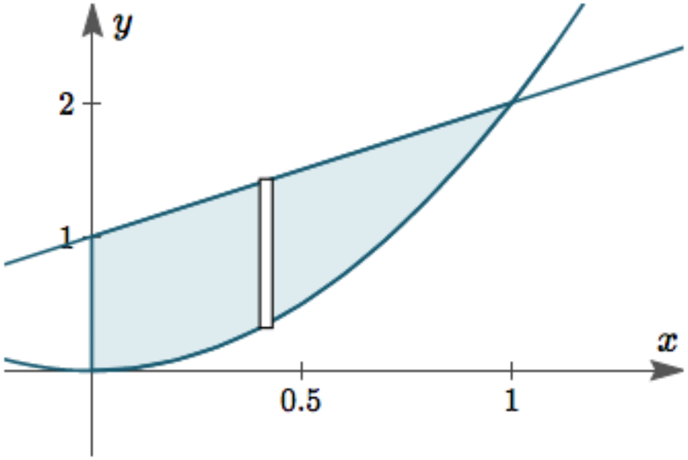
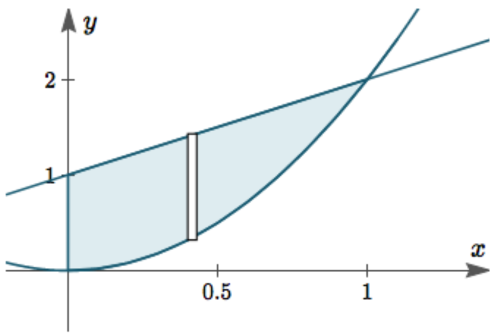
Solution: we sketch the upper and the lower bounding curves:



Area bounded by $y = 2x^2$ (the bottom curve), $y = x + 1$ (the line above), and $x = 0$, showing a typical rectangle.

Example

Solution:



Area bounded by $y = 2x^2$ (the bottom curve), $y = x + 1$ (the line above), and $x = 0$, showing a typical rectangle.

(source: *https://intmath.com*)

The lower limit of integration is $x = 0$ (since the question says $x \geq 0$).

Next, we need to find where the curves intersect so we know the upper limit of integration.

Example

Solution:

Equating the 2 expressions and solving:

$2x^2 = x + 1$

$2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$

The upper limit of the integration will be $x = 1$ (since we only need to consider $x \geq 0$. This is consistent with what we see in the picture).

So with $y_2 = x + 1$ and $y_1 = 2x^2$, the volume will be (next page).

Example

Solution:

Volume$= \pi \int_0^1 [(x+1)^2 - (2x^2)^2]\, dx$

$$= \pi \int_0^1 [(x^2+2x+1) - (4x^4)]\, dx$$

$$= \pi \left[\frac{x^3}{3} + x^2 + x - \frac{4x^5}{5}\right]_0^1$$

$$= \pi \left[\left(\frac{1}{3} + 1 + 1 - \frac{4}{5}\right) - (0)\right]$$

$$= \pi \left[\frac{5+30-12}{15}\right]$$

$$= \frac{23\pi}{15} = 4.817 \qquad \text{cm}^3$$

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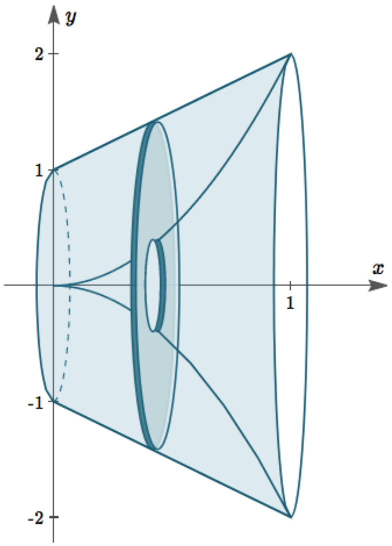
Example

Here's an illustration of the volume we have found. A typical "washer" with outer radius $y_2 = x + 1$ and inner radius $y_1 = 2x^2$ is shown.



The cup resulting from rotating the area bounded by $y = 2x^2$, $y = x + 1$, and $x = 0$ about the x-axis.

(source: *https://intmath.com*)

Exercises

1) Find the volume of an area rotated around the x-axis, bounded by a curve $y = x^2$ and the straight line $y = -x + 2$.
2) Find the volume of an area rotated around the x-axis, bounded by $y = 2x - x^2$ and $y = 0$.

Rotation around the y-axis

When the shaded area is rotated 360^o around the y-axis, the volume that is generated can be calculated by:

$$V = \pi \int_{c}^{d} x^2 \, dy$$

which means

$$V = \pi \int_{c}^{d} \{f(y)\}^2 \, dy$$

Where $x = f(y)$ is the equation of the curve **expressed in terms** of y, c and d are the upper and the lower y limits the area being rotated, and dy shows that the area is being rotated about the y-axis.

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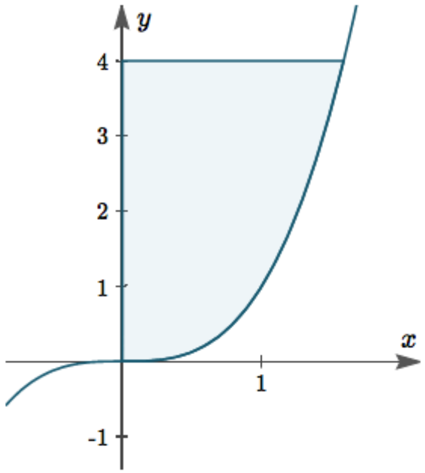
Example

Find the volume of the solid of revolution generated by rotating the curve $y = x^3$ between $y = 0$ and $y = 4$ about the y-axis.

Example

Find the volume of the solid of revolution generated by rotating the curve $y = x^3$ between $y = 0$ and $y = 4$ about the y-axis.

Solution: Here is the region we need to rotate

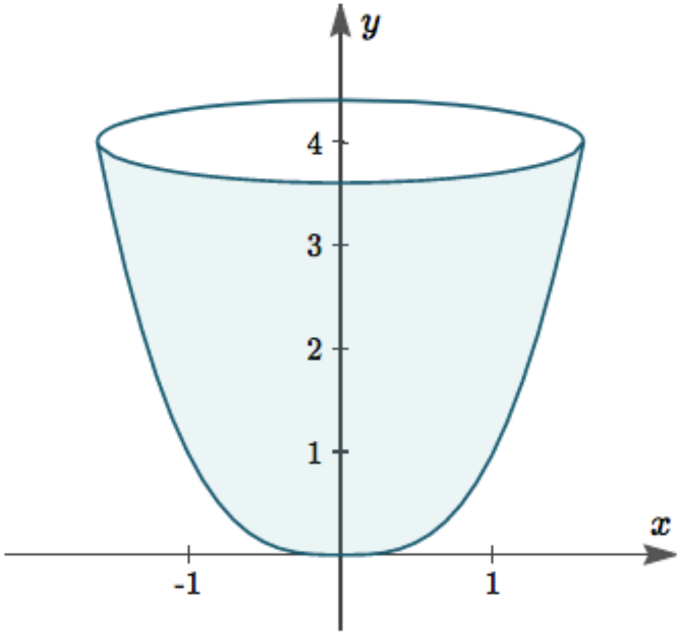


The graph of the area bounded by $y = x^3$, $x = 0$ and $y = 4$.

(source: *https://intmath.com*)

Example

Solution: And here is the volume generated when we rotate the region around the y-axis:



The volume generated when revolving the curve bounded by $y = x^3$, $x = 0$ and $y = 4$ around the y-axis.

(source: *https://intmath.com*)

Example

Solution: Firstly, express x in terms of y, so that we can apply the volume of solid of revolution formula. If $y = x^3$ then $x = y^{\frac{1}{3}}$.

The formula requires x^2, and on the squaring we have $x^2 = y^{\frac{2}{3}}$.

Vol$= \pi \int_c^d x^2 \, dy$

$$= \pi \int_0^4 y^{\frac{2}{3}} \, dy$$

$$= \pi \left[\frac{3y^{5/3}}{5}\right]_0^4$$

$$= \frac{3\pi}{5}\left[y^{5/3}\right]_0^4$$

$$= \frac{3\pi}{5}[10.079 - 0] = 19.0$$ units^3

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Exercises

1) Find the volume if the curve $y^2 = x$, bounded by $y = 4$ and $x = 0$, revolved about y-axis.
2) Find the volume if the curve $x^2 + 4y^2 = 4$ in quadrant I, revolved around the y-axis.
3) Find the volume if the curve $y^2 = x$, the limits of y-axis are $y = 0$ and $y = 2$, revolved around y-axis.
4) Find the volume of the area bounded by curves $y^2 = x$, the straight line $y = -x + 6$, and the x-axis. The area is revolved around the x-axis.

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Exercises (Applications)

1) A wine cask has a radius at the top of 30 cm and a radius at the middle of 40 cm. The height of the cask is 1 m. What is the volume of the cask (in L), assuming that the shape of the sides is parabolic?
2) A watermelon has an ellipsoidal shape with major axis 28 cm and minor axis 25 cm. Find its volume.

Shell Method (Method of Cylindears)

Another way to calculate volumes of revolution is the cylindrical shell method. This method is useful when the washer method is too difficult to carry out, usually because the inner and outer radius of the washer are awkward to express.

Shell Method (Method of Cylindears)

The formula for finding the volume of a solid of revolution using Shell Method is given by:

$$V = 2\pi \int_a^b r f(r)\, dr$$

where r is the radius from the center of rotation for a "typical" shell.

We will derive this formula a bit later, but first, let's start with some reminders.

Volume of a Hollow Cylinder

Example of hollow cylinders:

(source: *https://intmath.com*)

The volume of the material needed to make such hollow cylinders is given by the following, where R is the radius of the outer cylinder, and r is the radius of the inner wall, h is the height of the cylinder:

$$V = outer\ volume - hole\ volume$$

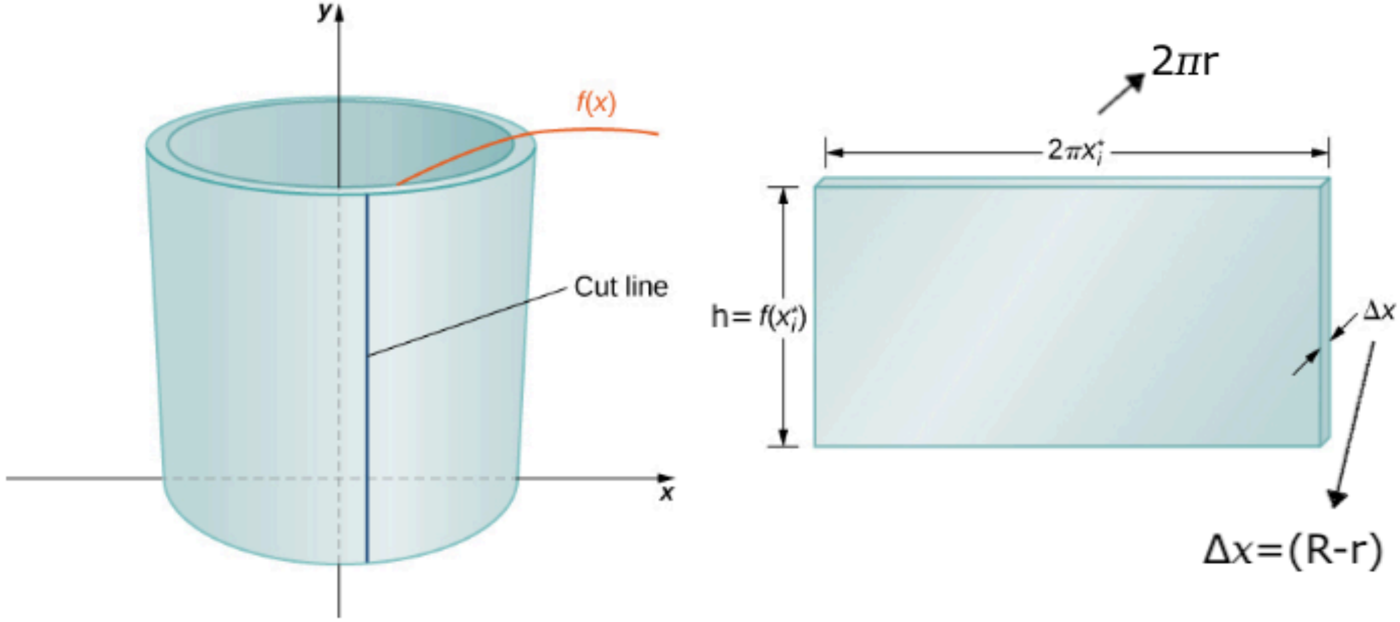
$$= \pi R^2 h - \pi r^2 h$$

$$= \pi h(R^2 - r^2)$$

Volume of a Hollow Cylinder

Another way to go about it would be to cut the cylinder vertically and lay it out flat. The width of the flat piece would be $2\pi r$, the circumference of the cylinder, and the thickness is $(R - r)$. The volume would be given by approximately:

$$V = 2\pi r \times h \times (R - r)$$



Example: Wedding Ring

A wedding ring has outer radius 10 mm and inner radius 8 mm. Its height (when laying flat on a table) is 3 mm. Find the volume using both methods mentioned previously.

Solution:

The volume is given by

a) Difference method

$$V = \pi h(R^2 - r^2) = \pi(3)(10^2 - 8^2) = 108\pi = 339.3 \quad \text{mm}^3$$

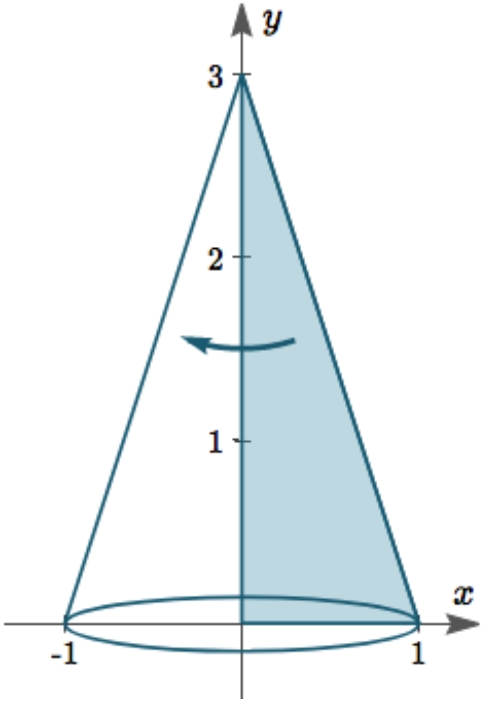
b) Laying out flat

$$V \approx 2\pi r \times h \times (R - r) = 2\pi(10)(3)(2) = 120\pi = 377.0 \quad \text{mm}^3$$

The second one is an approximation, but provides a useful way of calculating volumes of solids of revolution.

Rotation around the y-axis (Example: Cone)

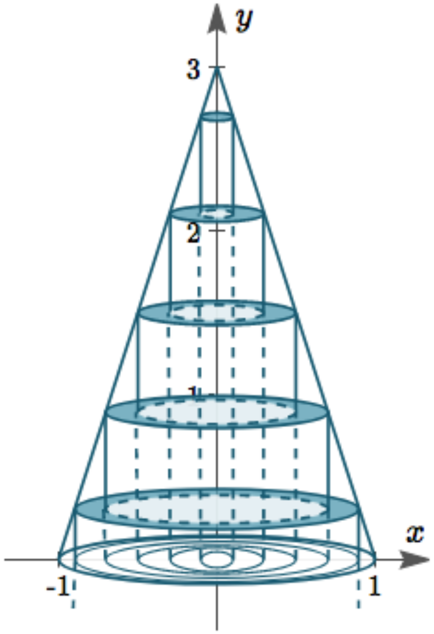
Consider rotating triangle bounded by $y = -3x + 3$ and the two axes, around the y-axis. It forms a cone:



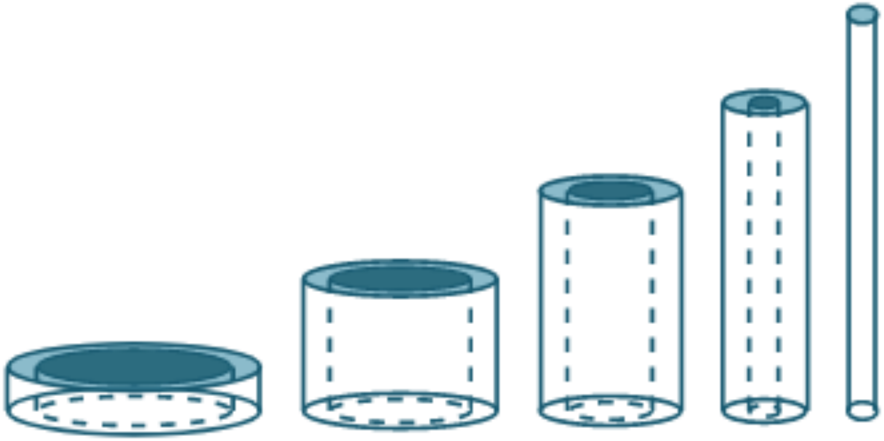
We can think of a solid of revolution as the sum of a set of hollow cylinders. We first, consider the simple case of cone, so that we can easily check our calculations.

Rotation around the y-axis (Example: Cone)

Slice the cylinder into a set of hollow cylinders, as follow:



Cone showing typical shells.



Typical shells that approximate our cone.

Rotation around the y-axis (Example: Cone)

As an example, we take one of the shells, cut it vertically, and lay it out flat.

Typical shell cut vertically and rolled out flat.

The volume of the above thin box shape is $l \times w \times h$.

The **length** (l) is given by $2\pi r$**.** The **height** is given by the function value for the particular shell, $f(r)$, and the **width** is the thicknessof the shell, which we write as $\Delta r = (R - r)$ (the change in r) or $\Delta x = (x_{i+1} - x_i)$ (if the curve is revolved y-axis). So the volume is: $V = 2\pi r \times f(r) \times \Delta r$

Rotation around the y-axis (Example: Cone)

Our 5 shells in the cone example are each 0.2 units thick. The height of each one is given by their appropriate function values, as follows:

$$f(0.1) = -3(0.1) + 3 = 2.7$$
$$f(0.3) = -3(0.3) + 3 = 2.1$$
$$f(0.5) = -3(0.5) + 3 = 1.5$$
$$f(0.7) = -3(0.7) + 3 = 0.9$$
$$f(0.9) = -3(0.9) + 3 = 0.3$$

So the total volume given by our five shells is:

sum$= 2\pi[(0.1)(2.7)(0.2) + (0.3)(2.1)(0.2) + (0.5)(1.5)(0.2) + (0.7)(0.9)(0.2) + (0.9)(0.3)(0.2)]$

$$= 2\pi(0.51) \approx 3.204$$

Check: the exact volume of a cone is given by

$$V = \frac{\pi r^2 h}{3} = \frac{\pi(1)^2(3)}{3} = \pi \approx 3.1416$$

Rotation around the y-axis (Example: Cone)

If we slice the cone such that there are many very thin shells, our value for the volume will get even closer to the real value.

In general we have,

$$V = \sum_{a}^{b} 2\pi r f(r) \Delta r$$

In this case of our cone, the rotation is around the y-axis, the **radius** is actually x (the distance from the center of rotation to the shell) and the **height** of each cylindrical shell is given by $f(x)$, which in this case is $f(x) = -3x + 3$.

As we take more and more thinner slices, Δx gets very small, and it tends to 0, our volume is given by the integral:

$$V = \int_0^1 2\pi x(-3x + 3)\, dx = \pi$$

General Formula for the Volume

Earlier, we saw this formula for the volume:

$$V = 2\pi \int_a^b rf(r)\, dr$$

The r changes depending on whether we are rotating around the x or y-axis. In the first example, we are rotating about the y-axis, so r is x (since the radius increases along the x-axis), and the height of each shell is given by $f(x)$.

So, when rotating around the y-axis, we write it as:

$$V = 2\pi \int_a^b xf(x)\, dx$$

The x represents the radius of a ”typical” shell, and $f(x)$ gives us the height of that typical shell.

Shell Method

Analogically, we will have several formula as follow:

1) If the area $A = \{(x, y): a \leq x \leq b, g(x) \leq y \leq f(x)\}$ is rotated around the y-axis, the volume will be $V_{oY} = 2\pi \int_a^b x(f(x) - g(x))\, dx$
2) If the area $A = \{(x, y): 0 \leq x \leq h(y), c \leq y \leq d\}$ is rotated around the x-axis, the volume will be $V_{oX} = 2\pi \int_c^d yh(y)\, dy$
3) If the area $A = \{(x, y): l(y) \leq x \leq h(y), c \leq y \leq d\}$ is rotated around the x-axis, the volume will be $V_{oX} = 2\pi \int_c^d y\big(h(y) - l(y)\big)\, dy$

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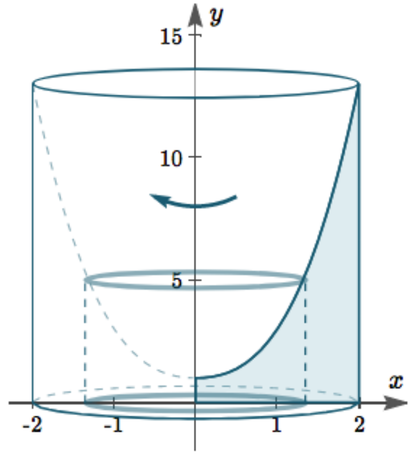
Example

Find the volume of the solid formed by rotating the area bounded by $y = x^3 + x^2 + 1$, the x and y-axis, and the line $x = 2$ around the y-axis.

Solution: the following is the sketch



The area between $y = x^3 + x^2 + 1$, the axes and $x = 2$ rotated round the y-axis.

(source: *https://intmath.com*)

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Example

Find the volume of the solid formed by rotating the area bounded by $y = x^3 + x^2 + 1$, the x and y-axis, and the line $x = 2$ around the y-axis.

Solution: Applying the formula for rotation around the y-axis, we have:

$$V = 2\pi \int_0^2 x(x^3 + x^2 + 1)\, dx$$

$$= 2\pi \int_0^2 (x^4 + x^3 + x)\, dx$$

$$= 2\pi \left[\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^2}{2}\right]_0^2$$

$$= \frac{124\pi}{5}$$

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Exercises

1) Find the volume of the solid of revolution if the area is in the quadrant I and bounded by x-axis, the curve $y = x^2$, and the straight line $y = -2x + 3$, the area is revolved around the x-axis.

2) Find the V_{oX} and V_{oY} for areas bellow:
   a) Between curves $y = x^4$ and $y = 2x - x^2$.
   b) Area in quadrant I, bounded by $y = 2x - x^2$, $x + y = 2$, and x-axis.

Volume of Solids of Revolution on Polar Coordinate System

Given a partition $P = \{\alpha = \theta_0, \theta_1, \dots, \theta_n = \beta\}$ on an interval $[\alpha, \beta]$.

For $i = 1, 2, \dots, n$, choose $\theta_i^* \in [\theta_{i-1}, \theta_i]$. Suppose a segment (juring) of the circle has an angle $\Delta\theta_i = \theta_i - \theta_{i-1}$ and a radius $r_i = f(\theta_i^*)$.

The arc of the segment on that circle will be

$$\frac{\Delta\theta_i}{2\pi} 2\pi r_i = r_i \Delta\theta_i$$

If the arc of the circle is revolved the x-axis, then it approach the surface (lateral) area of a cylinder, which can be computed by

$$2\pi y_i r_i \Delta\theta_i$$

where the radius is $y_i = r_i \sin\theta_i^*$, and the high is the arc of the segment on that circle, which is $r_i \Delta\theta_i$.

Volume of Solids of Revolution on Polar Coordinate System

If the partition is revolved around the x-axis, then we obtain a volume of solids revolution that approach the volume of a cone with high r_i and the base area (luas alas) is the surface (lateral) area (luas selimut) of the previous cylinder. So

$$V_i = \frac{1}{3}(2\pi y_i r_i \Delta\theta_i) r_i = \frac{2\pi}{3} r_i^3 \sin\theta_i^* \Delta\theta_i$$

Therefore, if an area D in polar coordinate is revolved around x-axis, then we have a volume of the solid of revolution which can be approached by

$$\frac{2\pi}{3}\sum_{i=1}^{n} r_i^3 \sin\theta_i^* \Delta\theta_i$$

Volume of Solids of Revolution on Polar Coordinate System

For $n \to \infty$ or $\max\{\Delta\theta_i : i = 1, 2, \dots, n\} \to 0$, we have a volume of solids of revolution

$$V_{oX} = \lim_{n\to\infty} \frac{2\pi}{3} \sum_{i=1}^{n} r_i^3 \sin\theta_i^* \Delta\theta_i$$
$$= \frac{2\pi}{3} \int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta$$

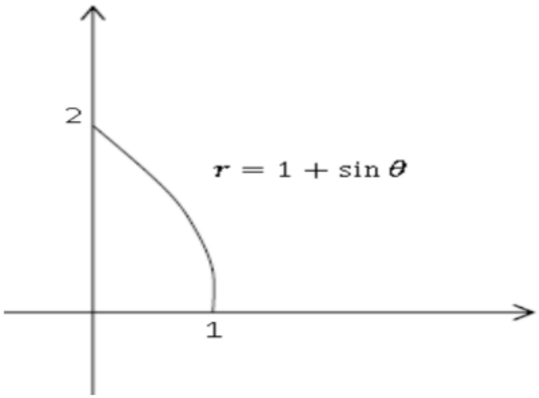
Analogically, if an area D in polar coordinate is revolving around the y-axis, then the volume of solids of revolution will be

$$V_{oY} = \frac{2\pi}{3} \int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta$$

Example 1

Given an area D inside the cardiode $r = 1 + \sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Find the volume of solids of revolution if the area D is revolved around y-axis.

Solution:

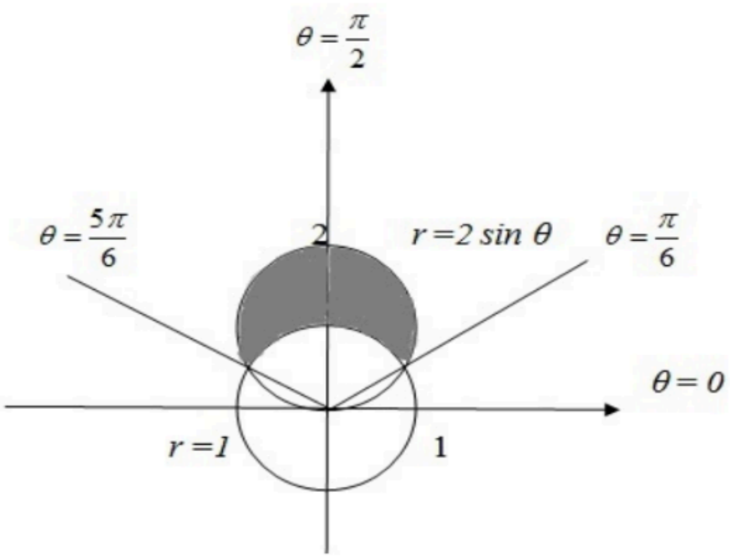


$$V_{oY} = \frac{2\pi}{3}\int_0^{\pi/2} (1 + \sin\theta)^3 \cos\theta \, d\theta$$

Example 2

An area D is inside the curve $r = 2\sin\theta$ and outside the curve $r = 1$. Find the volume of solid of revolution if the area of D is revolved around the x-axis.

Solution:



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Example 2

Solution:

The curve $r = 2\sin\theta$ and curve $r = 1$ are intersect at $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ and because of the area of D is symmetry with respect to y-axis then

$$V_{oX} = 2\left(\frac{2\pi}{3}\int_{\pi/6}^{\pi/2}\{(2\sin\theta)^3 - 1^3\}\sin\theta\, d\theta\right)$$

$$= \frac{4}{3}\pi\int_{\pi/6}^{\pi/2}(8\sin^4\theta - \sin\theta)\, d\theta = \frac{4}{3}\pi\int_{\pi/6}^{\pi/2}\left(8\left(\frac{1-\cos 2\theta}{2}\right)^2 - \sin\theta\right)d\theta$$

$$= \frac{4}{3}\pi\left[3\theta - 2\sin 2\theta + \frac{1}{4}\sin 4\theta + \cos\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{4}{3}\pi\left(\pi + \frac{3\sqrt{3}}{8}\right) = \frac{4}{3}\pi^2 + \frac{\sqrt{3}}{2}$$

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Exercises

1) Find the volume of solids of revolutions of an area D revolving the x or y-axis, if:
   a) D is an area in 1st quadrant and inside $r = 3\sin\theta$.
   b) D is an area inside the circle $r = 3\sin\theta$ and outside the cardiode $r = 1 + \sin\theta$
2) Find the volume of solids of revolutions of an area above the x-axis and inside $r = 4\cos\theta$ and inside $r = 2$ revolving the
   a) x-axis
   b) y-axis
3) Find the volume of solids of revolutions of an area inside the curve $r = \sin\theta$ and outside the curve $r = 1 - \cos\theta$ revolving around the y-axis.
4) Find the volume of solids of revolutions of an area inside $r = 2\cos 2\theta$ with $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{6}$ revolving around the y-axis.

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Thank You