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Applications of Integration: Pappus's Centroid Theorem (Pappus-Guldinus Theorem)

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Pappus-Guldinus Theorem

Pappus's theorem (also known as Pappus's centroid theorem, Pappus-Guldinus theorem or the Guldinus theorem) deals with the areas of surfaces revolution and with the volumes of solids of revolution.

The Pappus's theorem is actually two theorems that allow us to find surface areas and volumes without using integration.

1st Pappus's Theorem

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The first theorem of Pappus states that the surface area A of a surface of revolution obtained by rotating a plane curve C around a non-intersecting axis which lies in the same plane is equal to the product of the curve length L and the distance d traveled by the centroid of C:

$$A = Ld$$

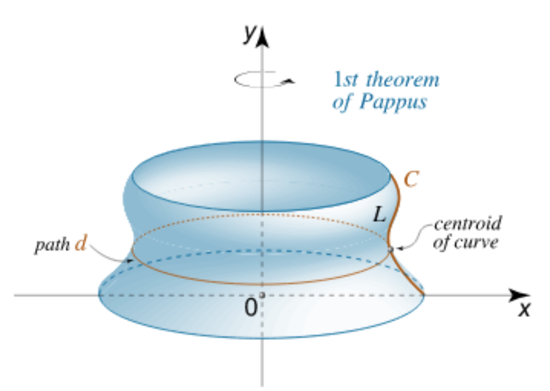


Figure 1.

Example 1

Find the area surface of revolution if the arc is a quarter circle $x^2 + y^2 = 1$ from $(0,1)$ to $(1,0)$ revolved around a straight line

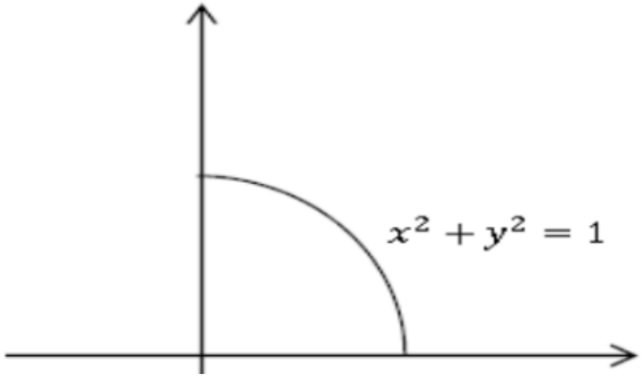
$a)$ $y = 4$

$b)$ $x + y = 2$

Solution:

The length of the arc is: $L = \frac{1}{4}(2\pi \cdot 1) = \frac{\pi}{2}$

Here is the figure of the arc



Example 1

Solution (continuation):

Suppose that the centroid (center of mass) of the arc is $(\bar{x}, \bar{y})$. If

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \frac{1}{\sqrt{1 - x^2}}\ dx$$

Then, we have

$$\bar{x} = \frac{\int_0^1 x\ ds}{\int_0^1 ds} = \frac{\int_0^1 \frac{x}{\sqrt{1-x^2}}\ dx}{\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx} = \frac{2}{\pi}$$

Since the arc is symmetric with respect to the curve $y = x$, then $\bar{y} = \bar{x}$. So the centroid is $\left(\frac{2}{\pi}, \frac{2}{\pi}\right)$.

Example 1

If A is the area of the surface of revolution then

a) For this case, the distance of centroid of the arc $\left(\frac{2}{\pi}, \frac{2}{\pi}\right)$ to the line $y = 4$ is

$r = \left|4 - \frac{2}{\pi}\right|$

so the area of the surface revolution is

$$A = L \cdot d = L \cdot (2\pi \cdot r) = \frac{\pi}{2} \cdot \left(2\pi \cdot \left|4 - \frac{2}{\pi}\right|\right) = 4\pi^2 - 2\pi$$

b) For this case, the distance of centroid of the arc $\left(\frac{2}{\pi}, \frac{2}{\pi}\right)$ to the line $x + y = 2$ is

2^{nd} Pappus's Theorem

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The second theorem of Pappus states that the volume of a solid of revolution obtained by rotating a lamina F about a non-intersecting axis lying in the same plane is equal to the product of the area A of the lamina F and the distance d traveled by the centroid of F:

$$V = Ad$$

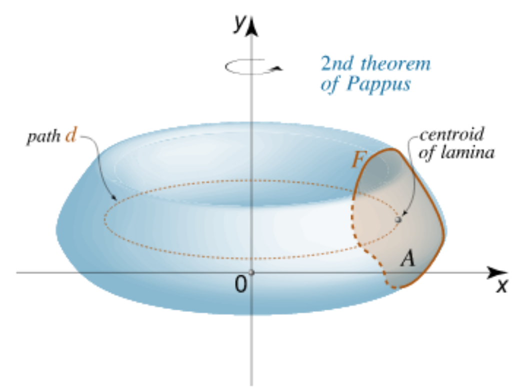

Figure 2.

Surface Area and Volume of a Torus

A torus is the solid of revolution obtained by rotating a circle about an external coplanar axis.

We can easily find the surface area of a torus using the 1st Theorem of Pappus. If the radius of the circle is r and the distance from the center of circle to the axis of revolution is R, then the surface area of the torus is

$$A = ld = 2\pi r \cdot 2\pi R = 4\pi^2 rR$$

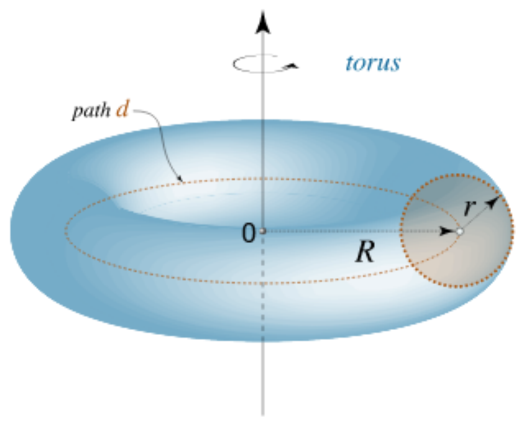

Figure 3.

Surface Area and Volume of a Torus

The volume inside the torus is given by the 2nd Theorem of Pappus:

$$V = Ad = \pi r^2 \cdot 2\pi R = 2\pi^2 r^2 R$$

The Pappus's theorem can also be used in reverse to find the centroid of a curve or figure.

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Thank You