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Pappus-Guldinus Theorem: Some Examples

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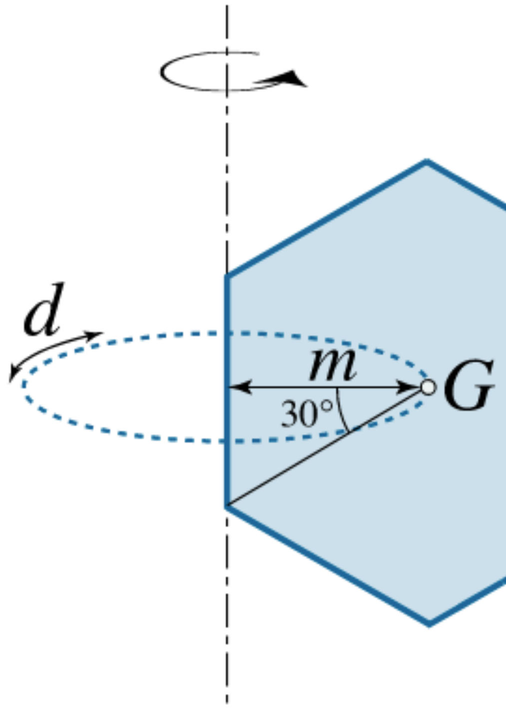
Example 1

A regular hexagon of side length a is rotated about one of the sides. Find the volume of the solid of revolution.

Solution:



Example 1

Solution (continuation):

Given the side of the hexagon a, we can easily find the apothem length m:

$$m = \frac{a}{2}\cot 30^o = \frac{a\sqrt{3}}{2}$$

Hence, the distance d traveled by the centroid C when rotating the hexagon is written in the form

$$d = 2\pi m = 2\pi \cdot \frac{a\sqrt{3}}{2} = \pi a\sqrt{3}$$

The area A of the hexagon is equal to

$$A = a^2\frac{3\sqrt{3}}{2}$$

Using the 2nd theorem of Pappus, we obtain the volume of the solid revolution:

$$V = A \cdot d = a^2\frac{3\sqrt{3}}{2} \cdot \pi a\sqrt{3} = \frac{9\pi a^3}{2}$$

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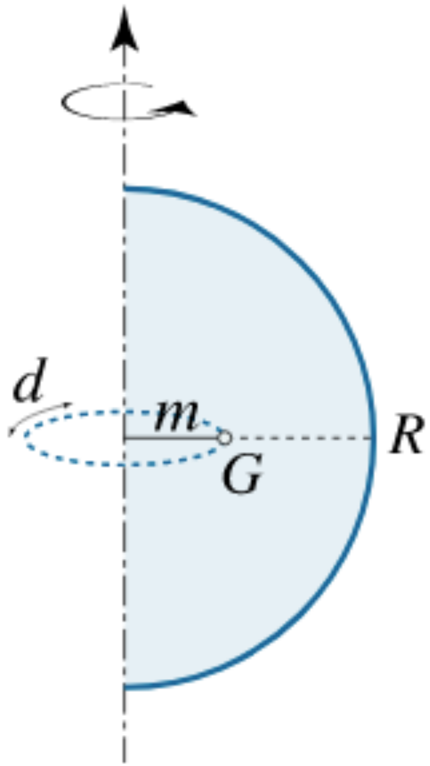
Example 2

Find the centroid of a uniform semicircle of radius R.

Solution:



Example 2

Solution (continuation):

Let m be the distance between the centroid G and the axis of rotation. When the semicircle makes the full turn, the path d traversed by the centroid is equal to

$$d = 2\pi m.$$

The solid of rotation is a ball of volume

$$V = \frac{4\pi R^3}{3}.$$

By the 2nd theorem of Pappus, we have the relationship

$$V = Ad,$$

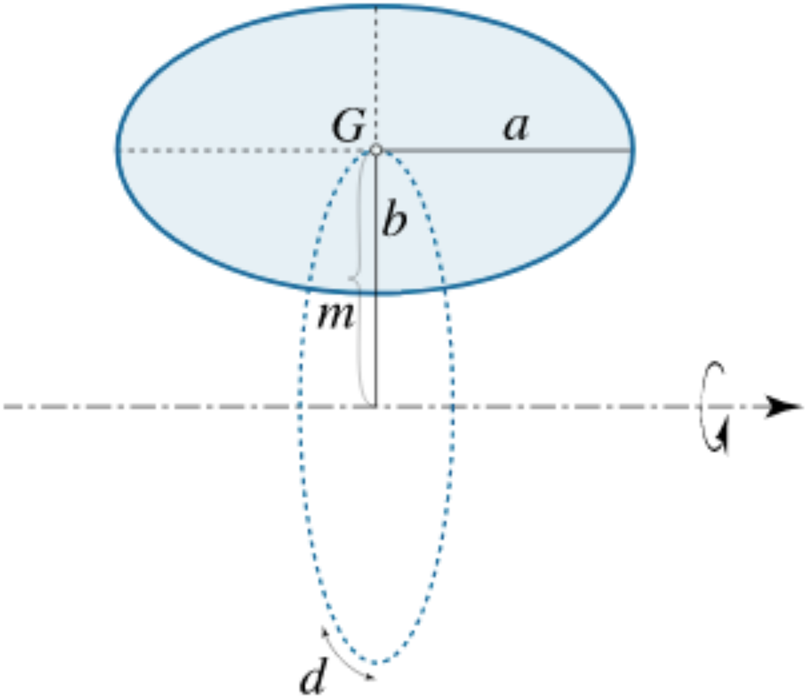
where $A = \frac{\pi R^2}{2}$ is the area of the semicircle. Hence,

$$m = \frac{V}{2\pi A} = \frac{\frac{4\pi R^3}{3}}{2\pi \cdot \frac{\pi R^2}{2}} = \frac{4R}{3\pi} \approx 0.42R$$

Example 3

An ellipse with the semimajor axis a and semiminor axis b is rotated about a straight line parallel to the axis a and spaced from it at a distance $m > b$. Find the volume of the solid of revolution.

Solution:



Example 3

Solution (continuation):

The volume of the solid of revolution can be determined using the 2nd theorem of Pappus:

$$V = Ad$$

The path d traversed in one turn by the centroid of the ellipse is equal to

$$d = 2\pi m$$

The area of the ellipse is given by the formula

$$A = \pi ab$$

Hence, the volume of the solid is

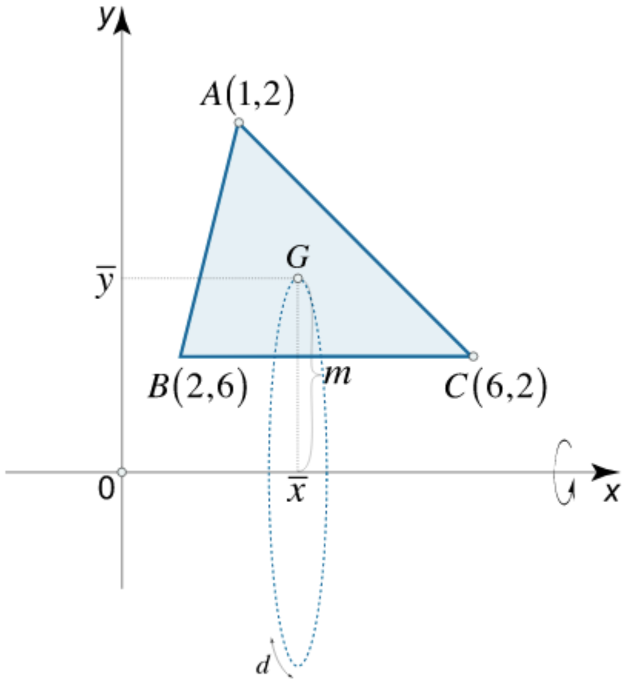
$$V = A \cdot d = \pi ab \cdot 2\pi m = 2\pi^2 mab$$

In particular, when $m = 2b$, the volume is equal to $V = 4\pi^2 ab^2$.

Example 4

A triangle with the vertices $A(1,2)$, $B(2,6)$, $C(6,2)$ is rotated about the x-axis. Find the volume of the solid of revolution thus obtained.

Solution:



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Example 4

Solution (continuation):

Since the coordinates of the vertices are known, we can easily find the area of the triangle. First we calculate the deterimnant:

$$\Delta = \begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 5 & 0 \end{vmatrix} = -20$$

Then the area of the triangle is

$$A = \frac{1}{2}|\Delta| = 10$$

Now we determine the centroid of the triangle:

$$\bar{x} = \frac{x_A + x_B + x_C}{3} = \frac{1+2+6}{3} = 3$$

$$\bar{y} = \frac{y_A + y_B + y_C}{3} = \frac{2+6+2}{3} = \frac{10}{3}$$

Example 4

Solution (continuation):

Thus, the centroid is

$$\left(3, \frac{10}{3}\right)$$

By the 2nd theorem of Pappus, the volume of the solid of revolution is given by

$$V = Ad = 2\pi m A$$

where $m = \bar{y}$ is the distance from the centroid G to the axis of rotation.

This yields:

$$V = 2\pi \cdot \frac{10}{3} \cdot 10 = \frac{200\pi}{3}$$

Thank You