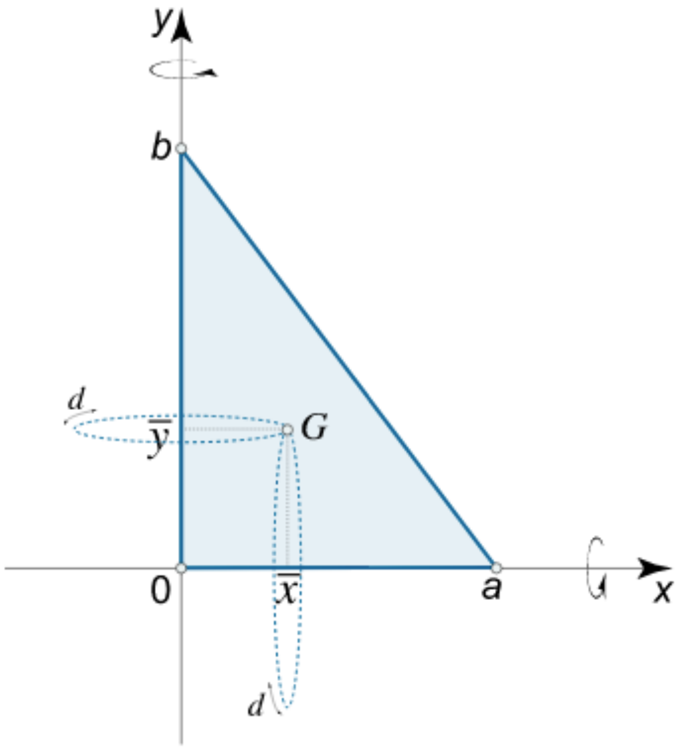
Pappus-Guldinus Theorem: Some Examples

Ivan Luthfi Ihwani, M. Sc.

Example 5

Find the centroid of a right triangle with legs a, b.

Solution:



UNIVERSITAS GADJAH MADA

Example 5

Solution (continuation):

To determine the coordinates of the centroid, we will use the 2nd theorem of Pappus.

Suppose first that the triangle is rotated about the y-axis. The volume of the obtained cone is given by

$$V_y = \frac{\pi a^2 b}{3}$$

The area of the triangle is

$$A = \frac{ab}{2}$$

Then, bu the Pappus's theorem,

$$V_y = 2\pi \bar{x} A, \qquad \Rightarrow \qquad \bar{x} = \frac{V_y}{2\pi A} = \frac{\frac{\pi a^2 b}{3}}{2\pi \cdot \frac{ab}{2}} = \frac{a}{3}$$

Example 5

Solution (continuation):

Let the triangle rotate now about the x-axis. Similarly, we find the volume

$$V_x = \frac{\pi a b^2}{3}$$

and the $\bar{y}$-coordinate of the centroid:

$$V_x = 2\pi \bar{y} A, \qquad \Rightarrow \qquad \bar{y} = \frac{V_x}{2\pi A} = \frac{\frac{\pi a b^2}{3}}{2\pi \cdot \frac{ab}{2}} = \frac{b}{3}$$

Thus, the centroid of the triangle is located at the point

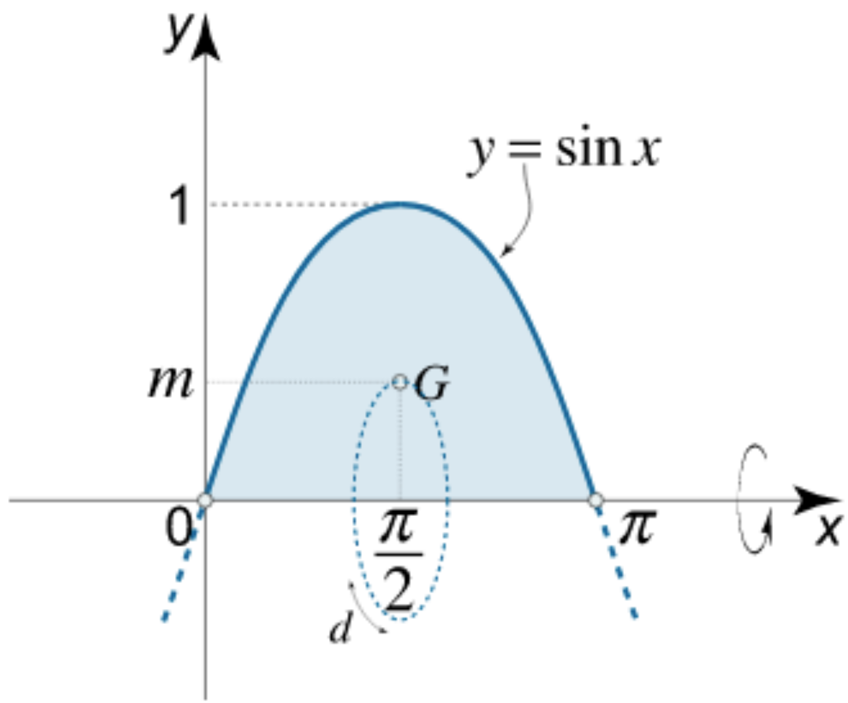
$$(\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{b}{3}\right)$$

which is the point of intersection of its median.

Example 6

Find the centroid of the region enclosed bu a half-wave of the sine curve and the x-axis.

Solution:



UNIVERSITAS GADJAH MADA

Example 6

Solution (continuation):

Let the point $(\bar{x}, \bar{y})$ denote the centroid of the figure. By symmetry, $\bar{x} = \frac{\pi}{2}$, so we need to calculate only the coordinate $\bar{y} = m$.

Using the disk method, we find the volume of the solid of revolution:

$$V = \pi \int_a^b f^2(x)\ dx = \pi \int_0^{\pi} \sin^2 x\ dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x)\ dx$$

$$= \frac{\pi}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi} = \frac{\pi^2}{2}$$

The area under the sine curve is

$$A = \int_a^b f(x)\, dx = \int_0^{\pi} \sin x\ dx = (-\cos x)|_0^{\pi} = -\cos \pi + \cos 0 = 2$$

Example 6

Solution (continuation):

The 2nd theorem of Pappus states that

$$V = Ad = 2\pi m A$$

Hence,

$$m = \frac{V}{2\pi A} = \frac{\frac{\pi^2}{2}}{4\pi} = \frac{\pi}{8} \approx 0.39$$

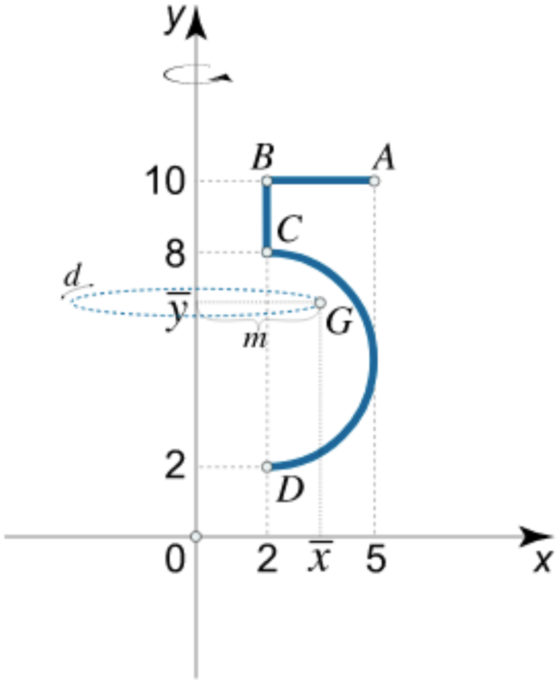
Thus, the centroid of the region has the coordinates

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$$

Example 7

A curve shown in Figure below is rotated about the y-axis. Find the area of the surface of revolution.

Solution:



UNIVERSITAS GADJAH MADA

Example 7

Solution (continuation):

We consider separately three sections of the curve and compute their centroids.

1) Horizontal line segment AB

The length is $L_{AB} = 3$. The centroid is located at the point $G_{AB} = (3.5, 10)$;

2) Vertical line segment BC

The length is $L_{BC} = 2$. The centroid is located at the point $G_{BC} = (2,9)$;

3) Semicircular arc CD.

The length is $L_{CD} = \pi R = 3\pi$. The centroid is located at the point $G_{CD} = (\bar{x}_{CD}, \bar{y}_{CD})$, where

$$\bar{x}_{CD} = 2 + \frac{2R}{\pi} = 2 + \frac{6}{\pi}; \qquad \bar{y}_{CD} = 5$$

Example 7

Solution (continuation):

Calculate the $\bar{x}$-coordinate of the centroif G of the whole curve:

$$\bar{x} = \frac{\bar{x}_{AB}L_{AB} + \bar{x}_{BC}L_{BC} + \bar{x}_{CD}L_{CD}}{L}$$

Where $L = L_{AB} + L_{BC} + L_{CD}$ is the total length of the curve.

By the 1st theorem of Pappus, the surface area is given by

$$A = Ld = 2\pi m L$$

Where d is the path traversed by the centroid of the curve in one turn and $m = \bar{x}$ is the distance from the centroid to the y-axis. Hence

$$A = 2\pi \cdot \frac{\bar{x}_{AB}L_{AB} + \bar{x}_{BC}L_{BC} + \bar{x}_{CD}L_{CD}}{L} \cdot L$$
$$= 310$$

UNIVERSITAS
GADJAH MADA

Thank You