Numerical Analysis I
Homework 6

Ivan L. Ihwani

Consider the system of nonlinear equations, $F(u_1, u_2, u_3) = 0$, where $F = (f_1, f_2, f_3)^T$ is defined as

$$\begin{cases} f_1(u_1, u_2, u_3) = 2u_1 - u_2 + \sinh(u_1) - 1, \\ f_2(u_1, u_2, u_3) = -u_1 + 2u_2 - u_3 + \sinh(u_2), \\ f_3(u_1, u_2, u_3) = -u_2 + 2u_3 + \sinh(u_3). \end{cases} \tag{1}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

1. All iterative methods learnt in this class can be extended to the nonlinear cases. Consider the nonlinear Jacobi iterative method which defined as:
   for $k = 0, 1, \ldots,$ until convergence
   solve $f_1\left(x, x_2^{(k)}, x_3^{(k)}\right)$ for $x = x_1^{(k+1)}$
   solve $f_2\left(x_1^{(k)}, y, x_3^{(k)}\right)$ for $y = x_2^{(k+1)}$
   solve $f_3\left(x_1^{(k)}, x_2^{(k)}, z\right)$ for $z = x_3^{(k+1)}$
   Suppose we use a fixed-point iterative method to solve each one-variable nonlinear equation. Consider $\mathbf{s}^{m+1} = \frac{1}{2}\sinh\left(\mathbf{s}^{(m)}\right) + \mathbf{c}$, where the constant $\mathbf{c}$ defined as $\mathbf{u}_2^{(k)} + 1$. Discuss its convergence.
2. Give a complete description of Newton's method for solving the nonlinear system given by (1), including input and output data, and stopping conditions. Write explicitly down the Jacobian matrices J, suppose zero initial vector is employed.
3. Discuss the special properties of the Jacobian matrices. Are the Jacobian matrices:
   (i) Symmetric Positive Definite?
   (ii) Diagonally Dominant?
4. Let $J^{(k)}$ be the Jacobian system at the k-th iteration.
   - What is $\left\|J^{(0)}\right\|_2$?
   - What is the condition number of $J^{(0)}$ in the 2-norm?
   - Suppose we want to solve $J^{(0)}\mathbf{x} = -F^{(0)}$, where the values in $J^{(0)}$ evaluations such that the relative error in $J^{(0)}$ is 0.00001. How large the relative error in x due to measurement error be? **Hint:** You may want to show $\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \Big/ \frac{\|\delta A\|}{\|A\|} \le \kappa(A)$ by considering $(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b}$. Here A is perturbed by infinitesimal δA and $\mathbf{x}$ must change by infinitesimal $\delta\mathbf{x}$.
5. Write down both Jacobi and Gauss-Seidel to solve each Jacobian system. Will these methods converges for some bounded approximation, $\left(\mathbf{u}_1^{(k)}, \mathbf{u}_2^{(k)}, \mathbf{u}_3^{(k)}\right)$. Provide any theorem to support your conclusions or provide a convergence analysis.

6. Consider the following iterative method

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \gamma \left(\mathbf{b} - J^{(0)}\mathbf{x}^{(k)}\right),$$

and $\gamma = \frac{2}{\rho\left(J^{(0)}\right)}$, where $\rho\left(J^{(0)}\right)$ is the spectral radius of $J^{(0)}$. Will the algorithm converge to the solution of $J^{(0)}\mathbf{x} = \mathbf{b}$ for any initial guess $\mathbf{x}^{(0)}$?
